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JOURNAL DEVOTED TO THE EXPERIMENTAL AND THEORETICAL STUDY OF THE FUNDAMENTAL CONSTITUENTS OF MATTER AND THEIR INTERACTIONS

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LAST NUMBER OF THIS VOLUME

S. KAMEFUCHI, L. O'RAIFEARTAIGH and ABDUS SALAM, Changes of variables and equivalence theorems in quantum field theories 529

R. A. ZDANIS, C. A. BRUNS, G. E. OWEN, L. MADANSKY and S. EDWARDS, Exchange stripping effects in the reaction $B^{11}(d, n)C^{12}_{4.43}$ 550

C. S. KHURANA and H. S. HANS, Cross-sections for (n, 2n) reactions at 14.8 MeV 560

G. B. BEARD and W. H. KELLY, "Self-scintillation" study of the beta decay of Rb^{87} 570

T. K. KRUEGER and B. MARGOLIS, The s-wave and p-wave strength functions 578

ROBERT R. JOHNSTON and KENNETH M. WATSON, The second-order optical potential for pion-nucleus scattering 583

WILLIAM W. PRATT, A survey of gamma ray branching ratios 598

E. L. CHURCH and J. WENESER, Anomalous electric-dipole internal conversion 602

PIERRE HILLION and JEAN-PIERRE VIGIER, Interaction theory in a new isobaric spin space 624

KARL WILDERMUTH and ROBERT L. CAROVILLANO, Optical giant resonances in nuclear reactions 636

S. M. SHAFROTH, The nucleus Y^{88} and the $Y^{88}-Sr^{88}$ mass difference 649

P. A. IONIDES, General equivalence theorem in vector field theory 662

TORSTEN GUSTAFSON, Angular distribution of photo-protons from deformed nuclei 665

Errata 681

News Items 682

Book Reviews 685

Author Index 695

Subject Index 699

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CHANGE OF VARIABLES AND EQUIVALENCE THEOREMS IN QUANTUM FIELD THEORIES

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Abstract: In a recent paper, Chisholm has proved the important theorem that the *S*-matrix in quantum field theories remains unchanged under any point transformation of field operators. In view of some unsatisfactory features in his argument, another proof of this theorem, within the framework of the conventional canonical formalism of field theories, is given. Further, it is pointed out that on the basis of Chisholm's theorem, most of the ordinary equivalence theorems in field theories can be obtained by rather trivial changes of variables in the Lagrangians. The equivalence theorem for the Yang-Mills field is discussed in detail.

1. Introduction

It is well-known in classical mechanics that the Lagrange equations are invariant under any transformation of the variables, which does not involve time derivatives (i.e. any *point* transformation). In quantum field theories, however, it is not yet known whether or not the whole theory remains invariant under similar transformations. In fact, so far the only kind of transformations which have been utilized to any great extent are the unitary (or canonical) transformations.

Recently, Chisholm [1]) has discussed this problem and claimed that any point transformation of variables does not alter the *S*-matrix, provided that the free field Lagrangian is kept unchanged under the transformation. His argument is based entirely on the asymptotic conditions for the field operators before and after the transformation. It seems to us, however, that the argument in his proof of the above theorem can be justified if and only if the so-called 'strong-convergence' of field operator at $t = \pm\infty$ is assumed. It is well-known from abstract field theory, however, that the strong convergence condition is too strict for interacting fields and has to be replaced by a 'weak convergence' condition. We therefore believe that while Chisholm's theorem

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is true as a general rule, his proof is not quite complete. In fact, if we apply Chisholm's argument, we are led, in some specially chosen cases, to a contradiction.

In view of this it seems worthwhile to re-examine this problem within the framework of the Heisenberg-Pauli formalism of field theories. We assume in addition, however, the adiabatic switching on and off of interactions, where by interactions we mean any product of *three* or more field operators, possibly all of the same field.

In sects. 2, 3 and 4, we shall try to show that for ordinary field theories, Chisholm's theorem is true, for general point transformations in the case of boson fields, and for a particular class of transformations in the case of fermion fields. Here, by ordinary field theories we mean those theories in which the interaction Lagrangians contain the time derivatives of boson fields at most in a linear form and contain no time derivatives of fermion fields. Although in carrying out our proof we shall have to impose a certain restriction on the transformation of fermion variables (which corresponds roughly to the 'point' nature of the transformations for boson fields), we may expect that the theorem would be true for more general transformations, since we know that it holds for unitary transformations which may in general contain time derivatives. It should be remarked, however, that as far as transformations which do not contain any time derivatives of field operators are concerned the above point transformations are more general than the unitary transformations. In other words, the point transformation and the unitary transformation are quite different things, neither class of transformation being entirely contained within the other.

A second problem which we shall discuss in this paper is that of equivalence theorems. In quantum field theories it has been known for some time that two (or more) alternative expressions for a Lagrangian or a Hamiltonian can lead to an identical final result for physically observable quantities. These relationships, called equivalence theorems, are usually proved by means of unitary transformations or by similar methods, which often involve rather complicated calculations. Once Chisholm's theorem is established, however, one sees that most of the equivalence theorems can be obtained simply by a change of variables in the corresponding Lagrangians.

In sect. 5, we shall re-consider the problem of equivalence theorems from this point of view, and, in particular, give a detailed discussion of the equivalence theorems for the Yang-Mills field with various types of interactions. In this way it is hoped that the connection between the equivalent interactions and the generalized gauge transformation will be made clear.

To keep the main trend of our argument continuous, some of the more mathematical parts of it have been relegated to an Appendix at the end of the paper.

2. Comment on Chisholm's Paper

Consider a Lagrangian for a system of fields $\phi_r(x)$, where $r = 1, 2, \ldots$:

$$\mathscr{L} = \mathscr{L}_0(\phi_1)+\mathscr{L}_0(\phi_2)+\ldots+\mathscr{L}_{\text{int}}(\phi_1\phi_2,\ldots), \tag{2.1}$$

and make the following change of variables:

$$\phi_r \to \phi'_r = \phi_r+F_r(\phi_1,\phi_2,\ldots)\ (r = 1, 2, \ldots), \tag{2.2}$$

where the functions F_r consist of terms which contain *at least two* field operators. Under this change of variables the bi-linear parts of $\mathscr{L}$, that is, the free field Lagrangians, remain unchanged, while the interaction Lagrangian changes, in general. Chisholm's theorem states that the S-matrix remains unchanged under the change of variables (2.2).

In order to prove this, he has shown that the field operators ϕ_r and ϕ'_r have the same asymptotic limits at $t = \pm\infty$, i.e.,

$$\phi^{(\alpha)}_{r\,\text{in(out)}} = \phi'^{(\alpha)}_{r\,\text{in(out)}}, \tag{2.3}$$

or, what is the same thing,

$$F^{(\alpha)}_{r\,\text{in(out)}} = 0. \tag{2.3'}$$

Here, the superscript (α) means the α-component of the operator, defined in the usual way by the scalar product of the operator and a normalizable wave packet $f^{(\alpha)}(x)$. To prove eq. (2.3') he has assumed the relation

$$F_{r\,\text{in(out)}} = \operatorname*{Lim}_{t=\pm\infty} F_r(\phi_1,\phi_2\ldots) = F_r(\phi_{1\,\text{in(out)}},\phi_{2\,\text{in(out)}},\ldots),$$

and then applied the Riemann-Lebesgue theorem to the last expression. Strictly speaking, however, such a procedure is allowed only when all field operators ϕ_r converge *strongly* † to their respective asymptotic limits. It is well-known, however, that the strong convergence is too strict a condition to be satisfied by any interacting fields and has to be replaced by *weak* convergence †. With weak convergence Chisholm's argument is no longer justified. To illustrate this we shall show that in certain cases his argument leads us to a contradiction.

We take, for example, a free neutral scalar field, the Lagrangian of which is given by

$$\mathscr{L} = \mathscr{L}_0 = -\tfrac{1}{2}(\partial_\mu\phi\partial_\mu\phi+\mu^2\phi^2), \tag{2.4}$$

and introduce a new variable ϕ' by

$$\phi = \phi'+f\phi'^2, \tag{2.5}$$

where f is a (coupling) constant. In terms of the new variable $\mathscr{L}$ takes the

† See, for example, ref. [2]) and papers quoted there.

following form †:

$$\mathscr{L} = \mathscr{L}_0(\phi') + \mathscr{L}_{\text{int}}(\phi'),$$
$$\mathscr{L}_{\text{int}}(\phi') = f\phi'^2(\Box - \mu^2)\phi' + \tfrac{1}{2}f^2\phi'^2(\Box - \mu^2)\phi'^2. \tag{2.6}$$

Now, according to Chisholm's argument eq. (2.3) or (2.3′) we get

$$f\phi'^{2(\alpha)} \to 0 \quad \text{(for all } \alpha\text{)} \quad \text{as} \quad t \to \pm\infty, \tag{2.7}$$

which means that

$$f\phi'^2 \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{2.7'}$$

This result cannot be corect for the following reason. According to Chisholm $P_\mu(\phi')$, the energy-momentum vector, should tend to $P_\mu(\phi'_{\text{in(out)}})$, which, in turn, should tend to $P_\mu^{(0)}(\phi'_{\text{in(out)}})$, the energy-momentum vector for the free field $\phi'_{\text{in(out)}}$. However, eq. (2.7′) indicates that not only the interaction term, but also the mass term of $\mathscr{L}_0(\phi')$ vanishes. Since the field $\phi'_{\text{in(out)}}$ should have the same mass as $\phi_{\text{in(out)}}$, we arrive at a contradiction.

In this connection it is worthwhile to notice that the appearance of such an inconsistency might also be reduced to the fact that there exists a domain in which the Riemann-Lebesque theorem cannot be properly applied. For example, in Chisholm's equation (2.7) one can choose for the wave functions $f^{(\alpha)}$, a set of plane waves $V^{-\frac{1}{2}}e^{i\mathbf{k}^{(\alpha)}\cdot\mathbf{x}}$, while $f^{(\beta)}, f^{(\gamma)}, \ldots$ are taken as normalizable wave packets. We may apply the Riemann-Lebesque theorem to Chisholm's equation (2.7) with such $f^{(\alpha)}$, $f^{(\beta)}$, $f^{(\gamma)}, \ldots$, but only for $\alpha \neq 0$, or $\mathbf{k}^{(\alpha)} \neq 0$, as can be checked. For $\alpha = 0$ or $k^{(\alpha)} = 0$, the Riemann-Lebesgue theorem does not apply. That is, the above eq. (2.7) or (2.7′) is true only for $\alpha \neq 0$. It is true that an argument based on such $f^{(\alpha)}$ is dangerous, because, strictly speaking, the set $f^{(\alpha)}$ is not normalizable. Nevertheless, it is interesting to see that with such a set and taking into account the fact that the Riemann-Lebesgue theorem does not apply for $\alpha = 0$, we can actually get the correct answer $P_\mu(\phi'_{\text{in(out)}}) = P_\mu^{(0)}(\phi'_{\text{in(out)}})$. This is because $P_\mu^{(0)}(\phi'_{\text{in(out)}})$ consists *only* of $P_\mu(\phi'_{\text{in(out)}})^{(\alpha)}$ with $\alpha = 0$ (all the components with $\alpha \neq 0$ vanish).

In view of the rather delicate nature of the asymptotic condition in abstract field theory (exemplified by our above results) we shall try, in the following section, to construct a proof of Chisholm's theorem, which lies within the older, Hamiltonian, formalism.

3. Invariance under Point Transformations in the Heisenberg-Pauli Formalism

3.1. PRELIMINARY CONSIDERATION

In this section we shall work exclusively in the Heisenberg representation Our aim will be to establish the invariance of the Heisenberg-Pauli formalism [4])

† To deal with such a system as eq. (2.6) with high derivatives one should have recourse, for example, to the general method proposed by Umezawa and Takahashi [3]).

under point transformations. To illustrate the idea involved we shall indicate first how this invariance is established in the case of classical fields. Let $\phi_r(x)$, $r = 1, 2, \ldots n$, be a set of n classical fields, whose motions are determined by a Lagrangian density $\mathscr{L}(\phi_r, \dot{\phi}_r)$ involving only the ϕ_r and their first time derivatives. The equations of motion are

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\phi}_r} = \frac{\delta \mathscr{L}}{\delta \phi_r}. \tag{3.1}$$

We now make a point transformation to new variables

$$\phi'_s(x) = \phi'_s(\phi_r(x)), \qquad \phi_r(x) = \phi_r(\phi'_s(x)), \tag{3.2}$$

i.e., any transformation (whose Jacobian is not zero) such that the ϕ' involve the ϕ but not the $\dot{\phi}$. It is then required to show that

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\phi}'_s} = \frac{\delta \mathscr{L}}{\delta \phi'_s}, \tag{3.3}$$

$$T_{\mu\nu} \equiv \sum_r \frac{\partial \mathscr{L}}{\partial \phi_{r,\mu}} \phi_{r,\nu} = T'_{\mu\nu} \equiv \sum_s \frac{\partial \mathscr{L}}{\partial \phi'_{s,\mu}} \phi'_{s,\nu}. \tag{3.4}$$

The proof is fairly simple:

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\phi}'_s} = \frac{\partial}{\partial t}\sum_r \frac{\partial \mathscr{L}}{\partial \dot{\phi}_r}\frac{\partial \dot{\phi}_r}{\partial \dot{\phi}'_s} = \sum_r \left(\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\phi}_r}\right)\frac{\partial \phi_r}{\partial \phi'_s} + \sum_r \frac{\partial \mathscr{L}}{\partial \dot{\phi}_r}\frac{\partial}{\partial t}\frac{\partial \phi_r}{\partial \phi'_s}$$

$$= \sum_r \frac{\delta \mathscr{L}}{\delta \phi_r}\frac{\partial \phi_r}{\partial \phi'_s} + \sum_r \frac{\partial \mathscr{L}}{\partial \dot{\phi}_r}\frac{\partial \dot{\phi}_r}{\partial \phi'_s} = \frac{\delta \mathscr{L}}{\delta \phi'_s},$$

as required. Note that we have used the relations

$$\frac{\partial \dot{\phi}_r}{\partial \dot{\phi}'_s} = \frac{\partial \phi_r}{\partial \phi'_s} \qquad \text{and} \qquad \frac{\partial}{\partial t}\frac{\partial \phi_r}{\partial \phi'_s} = \frac{\partial \dot{\phi}_r}{\partial \phi'_s}, \tag{3.5}$$

which can easily be verified from (3.2). Similarly, we have

$$T'_{\mu\nu} \equiv \sum_s \frac{\partial \mathscr{L}}{\partial \phi'_{s,\mu}} \phi'_{s,\nu} = \sum_s \sum_r \frac{\partial \mathscr{L}}{\partial \phi_{r,\mu}} \frac{\partial \phi_{r,\mu}}{\partial \phi'_{s,\mu}} \phi'_{s,\nu}$$

$$= \sum_r \frac{\partial \mathscr{L}}{\partial \phi_{r,\mu}} \sum_s \frac{\partial \phi_r}{\partial \phi'_s} \phi'_{s,\nu} = \sum_r \frac{\partial \mathscr{L}}{\partial \phi_{r,\mu}} \phi_{r,\nu} \equiv T_{\mu\nu}.$$

In this way the invariance of the theory under point transformations is established in the classical theory. As is seen, what is meant here by 'invariance' is this: if we start with one set of variables ϕ and use the Lagrangian formalism we arrive at the same result as if we first changed the variables ϕ to new variables ϕ' in the Lagrangian, then applied the Lagrangian formalism to the

ϕ', and afterwards expressed the results (field equations, etc.) in terms of the original ϕ. In other words, the field equations, etc., for the ϕ and the ϕ' are different in form but equal in content.

Let us now consider the question of invariance in this sense in quantum field theory. In this case we find that the situation is not so simple. If one attempts a proof analogous to the above one encounters two difficulties. The first is that due to the non-commutativity of the ϕ_r and $\dot{\phi}_r$, the direct proofs just given for the classical case break down (for example, it is no longer true that $\partial\mathscr{L}/\partial\dot{\phi}'_s = \sum_r(\partial\mathscr{L}/\partial\dot{\phi}_r)\ (\partial\dot{\phi}_r/\partial\dot{\phi}'_s)$). The second difficulty is that we must establish, in addition to (3.3) and (3.4), the further invariance

$$\left[\frac{\partial\mathscr{L}}{\partial\dot{\phi}_r}, \phi_s\right] = \left[\frac{\partial\mathscr{L}}{\partial\dot{\phi}'_r}, \phi'_s\right] = \Delta\delta_{rs}, \qquad \text{etc.,} \tag{3.6}$$

and

$$[P_\mu, \phi_r] = i\phi_{r,\mu}, \qquad [P'_\mu, \phi'_s] = i\phi'_{s,\mu}, \tag{3.6}$$

where Δ is a (singular) c-number.

It is clear, therefore, that the problem in quantum field theory is more complex than in classical theory. We consider it for boson fields and fermion fields separately.

3.2. BOSON FIELDS

First, we replace the general point transformation (3.2) of the classical theory by transformations

$$\phi_r = \phi_r(\phi'_s), \qquad \phi'_s = \phi'_s(\phi_r), \tag{3.7}$$

such that ϕ_r can be expressed as a power series in terms of the ϕ'_s and vice versa. (This is a familiar restriction on functions in quantum mechanics as well as in quantum field theory. An example of such a transformation is

$$\phi = \phi' - \phi'^2 + \ldots + (-1)^{n-1}\phi'^n + \ldots, \qquad \phi' = \phi + \phi^2 + \ldots + \phi^n + \ldots).$$

We define point transformations of boson variables to be transformations of this kind only.

Due to the commutativity of the ϕ_r a first result of eqs. (3.7) is that for $\phi''_k = \phi''_k(\phi'_s)$ we get

$$\frac{\partial\phi_r}{\partial\phi''_k} = \sum_s \frac{\partial\phi_r}{\partial\phi'_s}\frac{\partial\phi'_s}{\partial\phi''_k}. \tag{3.8}$$

We now consider the general definition of derivatives with respect to field variables. Let $F(\phi_r, \dot{\phi}_r)$ be any function of the ϕ_r, $\dot{\phi}_r$ which can be expressed as a (possibly infinite) sum of simple products of the quantities, e.g. $F = \phi + (\phi\dot{\phi} + \dot{\phi}\phi) + \phi\dot{\phi}^2$. Since $[\phi_r, \dot{\phi}_r] \neq 0$, the question is, what do we mean by

$\partial F/\partial\phi_r$ and $\partial F/\partial\dot{\phi}_r$? We define these quantities in three steps, in agreement with the original definition of Heisenberg and Pauli [4]):

Step 1:

$$\frac{\partial\phi_r}{\partial\phi_s}=\frac{\partial\dot{\phi}_r}{\partial\dot{\phi}_s}=\delta_{rs}, \qquad \frac{\partial\phi_r}{\partial\dot{\phi}_s}=\frac{\partial\dot{\phi}_r}{\partial\phi_s}=0. \tag{3.9}$$

Step 2:

Let X or Y or Z denote ϕ_r or $\dot{\phi}_r$ ($r=1, 2, \ldots, n$). Then we put

$$\frac{\partial}{\partial Z}XY=\frac{\partial X}{\partial Z}Y+X\frac{\partial Y}{\partial Z}, \tag{3.10}$$

with the generalization, for $X_i=\phi_r$ or $\dot{\phi}_r$ ($r=1, 2, \ldots, n$),

$$\frac{\partial}{\partial Z}\prod_{i=1}^{N}X_i=\sum_{j=1}^{N}\prod_{i=1}^{j-1}X_i\frac{\partial X_j}{\partial Z}\prod_{i=j+1}^{N}X_i. \tag{3.11}$$

Step 3:

If F_m is a simple product as in Step 2 and $F=\sum_{m=1}^{M}F_m$, we put

$$\frac{\partial}{\partial Z}F=\sum_{m=1}^{M}\frac{\partial F_m}{\partial Z}. \tag{3.12}$$

In this way the derivatives with respect to ϕ_r, $\dot{\phi}_r$ are unambiguously defined for functions $F(\phi_r, \dot{\phi}_r)$ of the kind mentioned (these are the only ones which we shall consider).

We are now in a position to examine the invariance of the Heisenberg-Pauli formalism under point transformations. Before proceeding to the investigation, however, it is useful to state two important theorems which follow directly form our definitions and which are proved in the Appendix. They are

$$\frac{\partial}{\partial\phi'_s}\frac{\partial}{\partial t}\phi_r(\phi'_s)=\frac{\partial}{\partial\phi'_s}\dot{\phi}_r(\phi'_s) \qquad \text{(Theorem I)},$$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial\phi'_s}\phi_r \qquad =\frac{\partial}{\partial\phi'_s}\dot{\phi}_r \qquad \text{(Theorem II)}.$$

It will be noticed that these are just the relations (3.5) which played such a vital role in the proof of invariance for the classical fields. We proceed now to the proof of invariance.

Let ϕ_r ($r=1, 2, \ldots, n$) be a set of n boson fields with Lagrangian density

$$\mathcal{L}=\sum_{r=1}^{n}\tfrac{1}{2}\dot{\phi}_r^2+G(\phi_r). \tag{3.13}$$

where G is any function (expressible as a power series) of the ϕ_r. The Heisenberg

Pauli formalism

$$\left[\frac{\partial \mathscr{L}}{\partial \dot{\phi}_r}, \phi_s\right] = \Delta\delta_{rs}, \qquad \frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\phi}_r} = \frac{\delta \mathscr{L}}{\delta \phi_r},$$
$$T_{\mu\nu} = \tfrac{1}{2}\sum_q \left\{\phi_{q,\nu}\frac{\partial \mathscr{L}}{\partial \phi_{q,\mu}} + \frac{\partial \mathscr{L}}{\partial \phi_{q,\mu}}\phi_{q,\nu}\right\} - \delta_{\mu\nu}\mathscr{L}, \qquad [P_\mu, \phi_r] = i\phi_{r,\mu}, \tag{3.14}$$

is equivalent in this case to

$$[\dot{\phi}_r, \phi_s] = \Delta\delta_{rs}, \qquad \ddot{\phi}_r = \frac{\delta G}{\delta \phi_r}, \qquad T_{44} = \sum_q \tfrac{1}{2}\dot{\phi}_q{}^2 - G(\phi_r), \quad \text{etc.} \tag{3.15}$$

Now let $\phi'_s = \phi'_s(\phi_r)$ (point transformation); then one has

$$\left[\frac{\partial \mathscr{L}}{\partial \dot{\phi}'_s}, \phi'_r\right] = \Delta\delta_{rs}, \qquad \frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\phi}'_s} = \frac{\delta \mathscr{L}}{\delta \phi'_s}, \qquad T_{\mu\nu} = T'_{\mu\nu}. \tag{3.16}$$

We prove this in the following way. By definition and theorem I we have

$$\frac{\partial \mathscr{L}}{\partial \dot{\phi}'_s} = \tfrac{1}{2}\sum_q \left(\dot{\phi}_q \frac{\partial \dot{\phi}_q}{\partial \dot{\phi}'_s} + \frac{\partial \dot{\phi}_q}{\partial \dot{\phi}'_s}\dot{\phi}_q\right) = \tfrac{1}{2}\sum_q \left(\dot{\phi}_q \frac{\partial \phi_q}{\partial \phi'_s} + \frac{\partial \phi_q}{\partial \phi'_s}\dot{\phi}_q\right),$$

and therefore, using eqs. (3.15) and (3.8)

$$\left[\frac{\partial \mathscr{L}}{\partial \dot{\phi}'_s}, \phi'_r\right] = \tfrac{1}{2}\sum_q \left\{[\dot{\phi}_q, \phi'_r]\frac{\partial \phi_q}{\partial \phi'_s} + \frac{\partial \phi_q}{\partial \phi'_s}[\dot{\phi}_q, \phi'_r]\right\}$$
$$= \tfrac{1}{2}\sum_q \left\{\Delta \frac{\partial \phi'_r}{\partial \phi_q}\frac{\partial \phi_q}{\partial \phi'_s} + \Delta \frac{\partial \phi_q}{\partial \phi'_s}\frac{\partial \phi'_r}{\partial \phi_q}\right\} = \Delta \frac{\partial \phi'_r}{\partial \phi'_s} = \Delta\delta_{rs}.$$

Further, using eq. (3.15) and theorem II, we get

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\phi}'_s} = \tfrac{1}{2}\sum_q \left\{\ddot{\phi}_q \frac{\partial \phi_q}{\partial \phi'_s} + \frac{\partial \phi_q}{\partial \phi'_s}\ddot{\phi}_q + \dot{\phi}_q\left(\frac{\partial}{\partial t}\frac{\partial \phi_q}{\partial \phi'_s}\right) + \left(\frac{\partial}{\partial t}\frac{\partial \phi_q}{\partial \phi'_s}\right)\dot{\phi}_q\right\}$$
$$= \tfrac{1}{2}\sum_q \left\{\frac{\delta G}{\delta \phi_q}\frac{\partial \phi_q}{\partial \phi'_s} + \frac{\partial \phi_q}{\partial \phi'_s}\frac{\delta G}{\delta \phi_q} + \dot{\phi}_q \frac{\partial \dot{\phi}_q}{\partial \phi'_s} + \frac{\partial \dot{\phi}_q}{\partial \phi'_s}\dot{\phi}_q\right\}$$

and finally, by a slight extension of eq. (3.8) to include functional derivatives,

$$= \frac{\delta G}{\delta \phi'_s} + \frac{\partial}{\partial \phi'_s}\sum_q \tfrac{1}{2}\dot{\phi}_q{}^2 = \frac{\delta \mathscr{L}}{\delta \phi'_s}.$$

We have still to show that $T_{\mu\nu} = T'_{\mu\nu}$; proof has been relegated to the Appendix as it involves a slightly more general definition of $T_{\mu\nu}$ than the usual one, and this makes it rather lengthy. Finally, the invariance of $[P_\mu, \phi_r] = i\phi_{r,\mu}$ follows immediately from that of $T_{\mu\nu}$ and of the commutation rules.

We now discuss briefly some points in connection with the above proof.

First, we note that the invariance has been shown only for $\partial/\partial\phi_r$ and $\partial/\partial\dot{\phi}_r$ as defined above (Steps 1, 2, 3). Secondly we started from a very special Lagrangian, involving the $\dot{\phi}_r$ only separately, quadratically and with numerical coefficients. As a matter of fact, the proof also holds good (as can easily be checked) if G involves the $\dot{\phi}_r$ in a linear way. However, it is not so certain (and in fact it seems to us at present to be unlikely) that the proof holds if G involves either $\dot{\phi}_r^2 f(\phi_r)$ or $\ddot{\phi}_r$. Thus the above proof is, so to speak, irreversible: We start with a standard Lagrangian (3.13), build another more general one (e.g. involving terms like $\dot{\phi}_r^2 f(\phi_r)$) by means of a point transformation, and then show the equivalence of the two. But, given some general Lagrangian (involving $\dot{\phi}_r^2 f(\phi_r)$, say) we have not shown that this is invariant under an arbitrary point transformation (much less have we shown that it can be reduced to the standard form).

In this connection we also observe the following point. For a Lagrangian transformed by an arbitrary point transformation, the transition from Lagrangian to Hamiltonian is not so simple in general because of the presence of terms bilinear in time derivatives. To express $\dot{\phi}'_r$ in terms of canonical variables we may have to have recourse to power series expansion in coupling constants. As for the calculation of the S-matrix in such a case a detailed discussion has been given in ref. [3]).

3.3. FERMION FIELDS

Much of the formalism for fermion fields is the same as in the boson case, and we therefore relegate the details to the Appendix and present here only the new features. The first new feature is the definition of 'point' transformations. Let ψ be a spinor. Then a point transformation of ψ is

$$\psi = \psi' F(\psi'), \tag{3.17}$$

where F is a function only of bilinear forms of ψ' and does not contain $\bar{\psi}'$. (this kind of term does not generally conserve the particle number). The property of F of being bilinear in ψ' is not a serious restriction as it is necessary in any case in order that ψ' transform like ψ under a Lorentz transformation. The restriction that F cannot contain $\bar{\psi}'$ is, on the contrary, a strong one and is analogous to the condition that ϕ'_s does not contain $\dot{\phi}_r$ in the boson case. Thus, neither in the boson nor in the fermion case does the class of point transformations include the general unitary transformation.

With this definition of point transformations, and with a standard Lagrangian

$$\mathscr{L} = -\tfrac{1}{2}\bar{\psi}(\gamma\overrightarrow{\partial}+m)\psi+\tfrac{1}{2}\bar{\psi}(\gamma\overleftarrow{\partial}-m)4+g\bar{\psi}(\gamma\text{'s; boson operators})\psi,$$

the proof of invariance proceeds analogously to the boson case. There is a slight change of signs in the definition of $\partial/\partial\psi_\alpha$, etc. due to the anti-commuta-

tion, but eq. (3.8) and theorems I and II hold as before. The transformation (3.17) can also be generalized to include boson variables in F.

To summarize, therefore, the general result of this section is that, with the definitions given, the Heisenberg-Pauli formalism is invariant under the point transformation

$$\psi = \psi' F(\psi', \phi'), \qquad \phi = \phi(\phi'), \tag{3.18}$$

F being bilinear in ψ'.

4. Proof of Equality of S-Matrices

In the preceding section we have shown that the Lagrangian-Hamiltonian formalism is invariant under the point transformation (3.18). This means that it does not matter which set of variables ϕ or ϕ' one may choose in describing a given physical system; it is merely a matter of convenience. Of course, the physical meanings to be attached to these variables will in general vary according to the choice of variables.

In quantum field theory, we usually choose the variables in such a way that the free field operators obey the equations of motion for a simple harmonic oscillator. This allows us to make the usual particle interpretation of fields. Thus, if we restrict ourselves to those variables which satisfy this requirement, the change of variables is to be limited to the following type:

$$\phi'_r(x) = \phi_r(x) + g_r F_r(g_s; \phi_s(x)), \qquad (r, s = 1, 2, \ldots, n), \tag{4.1}$$

where the F_r are functions of the ϕ, each being a sum of terms which are products of at least two field operators. For dimensional reasons, constants g_s are introduced here, which may be regarded as coupling constants, for in the Lagrangian they appear only in 'interaction' terms, by which we mean terms containing *three* or more field operators (possibly all of the same field).

Now, as was proved in eq. (3.16), we have for the total Hamiltonian (in the Schrödinger representation)

$$H = H_0(\phi) + H_{\text{int}}(\phi) = H' = H_0(\phi') + H'_{\text{int}}(\phi'). \tag{4.2}$$

Let us consider two interaction representations, in one of which H_{int}, and in the other of which H'_{int} appears in the Schrödinger equation, the respective state vectors being $\Psi(t)$ and $\Psi'(t)$. Since $\Psi(t)$ and $\Psi'(t)$ are connected with the common Schrödinger (or Heisenberg) representation by unitary transformations, they are mutually related by a unitary transformation $U(t)$,

$$\Psi(t) = U(t)\Psi'(t), \tag{4.3}$$

with, for example,

$$U(t) = \mathrm{e}^{-iH_0(\phi_{t=0})t}\mathrm{e}^{iH_0(\phi'_{t=0})t}.$$

It is important to notice here that since these two representations differ only by the terms depending on the interaction, they should coincide at $t = \pm\infty$, when both H_{int} and H'_{int} are switched off under the adiabatic hypothesis which, as we have said, we assume for all interactions (products of *three* or more operators) as in the ordinary theory. Thus, we have

$$U(t) \to 1 \qquad \text{as} \qquad t \to \pm\infty. \tag{4.4}$$

This can also be seen by rewriting $U(t)$ in the form

$$U(t) = \exp\{i[H'_{\text{int}} - H_{\text{int}}]t + \tfrac{1}{2}[H'_{\text{int}}, H_{\text{int}}]t^2 + \ldots\}.$$

It should be remarked here that the above requirement which implies "inter alia" that $\phi^n \to 0$, $n \geqq 3$) is weaker than Chisholm's asymptotic condition which, as we have seen, implied $\phi^2 \to 0$ (so that $\phi = \phi' + f\phi'^2 \to \phi'$) and so led to a contradiction with the physical requirement that the mass term in the Lagrangian remain non-zero for $t \to \pm\infty$ †.

Corresponding to the above two interaction representations we get two different expressions for the S-matrix, S and S', which can be obtained by taking H_{int} and H'_{int}, respectively, as interaction Hamiltonian. From (4.3) S and S' are connected by

$$S' = U^{-1}(+\infty)SU(-\infty),$$

and using (4.4) we now get

$$S' = S. \tag{4.5}$$

This means that two different interaction Hamiltonians $H_{\text{int}}(\phi)$ and $H'_{\text{int}}(\phi')$ lead to identical S-matrices, and this is exactly Chisholm's result (apart from the restrictions imposed on the transformations of fermion variables in our case). Finally, let us remark that while the Heisenberg operators ϕ and ϕ' are not necessarily connected by a unitary transformation (in this sense our class of point transformations is wider than the class of unitary transformations) the operators ϕ and ϕ' in the respective interaction representations are thus connected, because they satisfy the same commutation relations.

5. Equivalence Theorems

In quantum field theories there exist several kinds of theorems, called equivalence theorems, which state that two alternative expressions for the interaction Lagrangian or Hamiltonian lead to identical final answers for phys-

† Of course, our assumption that terms such as $\phi^n \to 0$, $n \geqq 3$ may not be quite free from objection from a strict mathematical point of view either, but we feel safer to make this assumption, since (in contrast to the assumption $\phi^2 \to 0$) it does not seem to lead to any physical contradiction. Note also that in our case, although $H_0(\phi) = H_0(\phi')$ at $t = \pm\infty$, this does not necessarily imply $\phi = \phi'$.

ically observable quantities. These theorems have so far been proved by means of unitary transformations or by similar methods, which are not in general simple.

However, when viewed from the standpoint of the theorem established above, the equivalence theorems are in most cases an immediate coesequence of simple changes of variables in the original Lagrangians. Almost trivial examples of this are the proofs of gauge invariance of quantum electrodynamics [5]), the equivalence theorems in meson theories discussed by Nelson, Dyson and Case [6]), and the equivalence theorem in neutral vector meson theory discussed recently by two of the present authors [7]).

To illustrate, let us take as an example the Lagrangian for the π-meson–nucleon system given by

$$\mathscr{L} = -\bar{\psi}(\gamma\partial+m)\psi-\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi}+\mu^2\boldsymbol{\phi}\cdot\boldsymbol{\phi})+ig\bar{\psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\psi, \tag{5.1}$$

and make the following change of variables:

$$\psi\to\psi'=\psi\exp\left[-\frac{ig}{2m}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right], \qquad \boldsymbol{\phi}\to\boldsymbol{\phi}'=\boldsymbol{\phi}, \tag{5.2}$$

which satisfy the conditions (3.18) and (4.1) for the F in our invariance theorem. Substituting (5.2) into (5.1) and omitting the prime one gets

$$\begin{aligned}\mathscr{L} &= -\bar{\psi}(\gamma\partial+m)\psi-\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi}+\mu^2\boldsymbol{\phi}\cdot\boldsymbol{\phi})\\ &-m\bar{\psi}\left\{\exp\left[\frac{ig}{m}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right]-1-\frac{ig}{m}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right\}\psi+\frac{ig}{m}\bar{\psi}\gamma_5\gamma_\mu\left[f\left(-g,\frac{\gamma_5\boldsymbol{\phi}}{m}\right)\partial_\mu\boldsymbol{\phi}\right]\cdot\boldsymbol{\tau}\psi,\end{aligned} \tag{5.3}$$

where $f(g, \mathbf{a})$ is the operator in iso-space introduced in a previous paper [8]) and given by

$$f(g,\mathbf{a})=\frac{\exp[g\alpha(\mathbf{a})]-1}{g\alpha(\mathbf{a})}, \tag{5.4}$$

with $\alpha(\mathbf{a})$ such that $\alpha(\mathbf{a})\mathbf{b}=\mathbf{b}\wedge\mathbf{a}$, where $\mathbf{b}$ is an arbitrary iso-vector. Eq. (5.3) is essentially the result obtained by Drell and Henly [9]), and when expanded in powers of g, it gives the familiar expression of Nelson, Dyson and Case.

The equivalence theorem in neutral vector meson theory with partial conservation law can be proved in a similar manner.

6. Equivalence Theorem for the Yang-Mills Field

6.1. THE FREE YANG-MILLS FIELD

The Lagrangian is given by †

$$\mathscr{L}_{\text{Y.M.}}=\mathscr{L}_{\text{Y.M.0}}(\mathbf{u}_\mu)+\mathscr{L}_{\text{Y.M.int}}(\mathbf{u}_\mu), \tag{6.1}$$

† In this section we use the same notation as in ref. [8]).

where

$$\begin{aligned}\mathscr{L}_{\text{Y.M.}\,0}\ (u_\mu) &= -\tfrac{1}{4}\mathbf{u}_{\mu\nu}\cdot\mathbf{u}_{\mu\nu}+\tfrac{1}{2}\kappa^2\mathbf{u}_\mu\cdot\mathbf{u}_\mu,\\ \mathscr{L}_{\text{Y.M. int}}(u_\mu) &= -\tfrac{1}{2}g\mathbf{u}_{\mu\nu}\cdot(\mathbf{u}_\mu\wedge\mathbf{u}_\nu)-\tfrac{1}{4}g^2(\mathbf{u}_\mu\wedge\mathbf{u}_\nu)\cdot(\mathbf{u}_\mu\wedge\mathbf{u}_\nu).\end{aligned} \tag{6.1'}$$

Eq. (6.1) is invariant except for the mass term, under the following transformation:

$$\mathbf{u}_\mu\to\mathbf{u}'_\mu = S(g,\Lambda)\mathbf{u}_\mu+f(g,\Lambda)\partial_\mu\Lambda, \tag{6.2}$$

where $S(g,\Lambda)$ is defined by

$$S(g,\Lambda) = \exp[g\alpha(\Lambda)] \tag{6.2'}$$

and $f(g,\Lambda)$ is the same operator as that defined in eq. (5.4) †.

In terms of the Stueckelberg variables eq. (6.1) becomes

$$\mathscr{L}_{\text{Y.M.}} = \mathscr{L}_{\text{Y.M.}\,0}(\mathbf{A}_\mu,\mathbf{B})+\mathscr{L}_{\text{Y.M. int}}(\mathbf{U}_\mu), \tag{6.3}$$

where

$$\begin{aligned}\mathscr{L}_{\text{Y.M.}\,0}(\mathbf{A}_\mu,\mathbf{B}) = &-\tfrac{1}{2}\partial_\mu\mathbf{A}_\nu\cdot\partial_\mu\mathbf{A}_\nu-\tfrac{1}{2}\kappa^2\mathbf{A}_\mu\mathbf{A}_\mu\\ &-\tfrac{1}{2}\partial_\mu\mathbf{B}\cdot\partial_\mu\mathbf{B}-\tfrac{1}{2}\kappa^2\mathbf{B}\cdot\mathbf{B}+\tfrac{1}{2}(\partial_\mu\mathbf{A}_\mu+\kappa\mathbf{B})^2,\end{aligned} \tag{6.3'}$$

$$\mathbf{U}_\mu = \mathbf{A}_\mu+\frac{1}{\kappa}\partial_\mu\mathbf{B}\equiv\mathbf{A}_\mu+\partial_\mu\mathbf{b}.$$

The transformation (6.2) now takes the following form:

$$\begin{aligned}\mathbf{A}_\mu\to\mathbf{A}'_\mu &= S(g,\Lambda)\mathbf{A}_\mu+[f(g,\Lambda)-1]\partial_\mu\Lambda,\\ \partial_\mu\mathbf{B}\to\partial_\mu\mathbf{B}' &= S(g,\Lambda)\partial_\mu\mathbf{B}+\kappa\partial_\mu\Lambda.\end{aligned} \tag{6.4}$$

Now, we can show that when making a change of variables, such as

$$\mathbf{A}_\mu\to\mathbf{A}'_\mu = S(g,\mathbf{b})\mathbf{A}_\mu+[f(g,\mathbf{b})-1]\partial_\mu\mathbf{b},\qquad \mathbf{B}\to\mathbf{B}'=\mathbf{B}, \tag{6.5}$$

or

$$\mathbf{U}_\mu\to\mathbf{U}'_\mu = S(g,\mathbf{b})\mathbf{A}_\mu+f(g,\mathbf{b})\partial_\mu\mathbf{b}\equiv\mathbf{W}_\mu, \tag{6.5'}$$

the Lagrangian is transformed into

$$\begin{aligned}\mathscr{L}'_{\text{Y.M.}} &= \mathscr{L}_{\text{Y.M.}\,0}(\mathbf{A}'_\mu,\mathbf{B}')+\mathscr{L}_{\text{Y.M. int}}(\mathbf{U}'_\mu)\\ &= \mathscr{L}_{\text{Y.M.}\,0}(\mathbf{A}_\mu,\mathbf{B})+\mathscr{L}_{\text{Y.M. int}}(\mathbf{A}_\mu)+\tfrac{1}{2}\kappa^2[\mathbf{U}_\mu\cdot\mathbf{U}_\mu-\mathbf{W}_\mu\cdot\mathbf{W}_\mu].\end{aligned} \tag{6.6}$$

The reason why the last term in eq. (6.6) appears can be understood in the following way. Since the change of variables (6.5'), $\mathbf{U}'_\mu\to\mathbf{A}_\mu$, is essentially of the same form as (6.2), $\mathbf{u}'_\mu\to\mathbf{u}_\mu$ (by replacing $\mathbf{u}_\mu$ and Λ in (6.2) by $\mathbf{A}_\mu$ and $\mathbf{b}$, respectively, one can get (6.5')), the non-gauge invariant term $\frac{1}{2}\kappa^2\mathbf{U}'_\mu\mathbf{U}'_\mu$ gives rise to $\frac{1}{2}\kappa^2\mathbf{W}_\mu\mathbf{W}_\mu$.

Eq. (6.6) is exactly the expression which was obtained before by means of a unitary transformation [8]). Thus, we can justify, a posteriori, the above change

† For detailed properties of the operators S and f, see eq. (26) of ref. [8]).

of variables (6.5) or (6.5′), although the invariance of the theory under this transformation is *not* guaranteed by our theorem because of the presence of the time derivative of **B** in the transformation of $\mathbf{A}_4$. This result seems to suggest that our theorem could be generalized to include such cases as well.

Generalizing the argument one can make the following statement: Suppose that a Lagrangian $\mathscr{L}$ is of the form

$$\mathscr{L} = \mathscr{L}_{\text{Y.M.}}(\mathbf{u}_\mu)+\mathscr{L}_1(\mathbf{u}_\mu, \phi)+\mathscr{L}_2(\mathbf{u}_\mu, \phi), \tag{6.7}$$

where the ϕ are field operators interacting with $\mathbf{u}_\mu$. The quantities $\mathscr{L}_1$ and $\mathscr{L}_2$ are invariant and non-invariant parts of $\mathscr{L}$ under the gauge transformation (6.2), combined with some appropriately defined transformations for the ϕ, which are needed to make $\mathscr{L}_{\text{Y.M.}}$ (mass-term) and $\mathscr{L}_1$ invariant. Furthermore. we shall assume that the latter gauge transformation satisfies the condition for point transformations of our theorem. Then, we can say that $\mathscr{L}$ is equivalent to the following expression:

$$\begin{aligned}\mathscr{L} = \mathscr{L}_{\text{Y.M.0}}(\mathbf{A}_\mu, \mathbf{B})+\mathscr{L}_{\text{Y.M. int}}(\mathbf{A}_\mu)+\tfrac{1}{2}\kappa^2[\mathbf{U}_\mu\cdot\mathbf{U}_\mu-\mathbf{W}_\mu\cdot\mathbf{W}_\mu]\\ +\mathscr{L}_1(\mathbf{A}_\mu, \phi)+\mathscr{L}_2(\mathbf{W}_\mu, \phi'),\end{aligned} \tag{6.7′}$$

where ϕ' is obtained from ϕ by making the gauge transformation with the function **b** (not with Λ). That eq. (6.7′) follows from (6.7) is due to the fact that the transformation to 'eliminate' the **b**–field is of the same form as the gauge transformation. In the following we shall consider some concrete examples.

6.2. GÜRSEY'S LAGRANGIAN

Following Gürsey [11]) we can write †

$$\mathscr{L}_{\text{G}} = -\tfrac{1}{4}\mathbf{u}_{\mu\nu}\cdot\mathbf{u}_{\mu\nu}-\tfrac{1}{2}\kappa^2\left[\mathbf{u}_\mu+\frac{1}{\kappa}f(-g, \phi/\kappa)\partial_\mu\phi\right]^2+\mathscr{L}_{\text{Y.M. int}}(\mathbf{u}_\mu). \tag{6.8}$$

One can easily check that (6.8) is *invariant* under the following transformation:

$$\begin{aligned}&\mathbf{u}_\mu \to \mathbf{u}'_\mu = S(g, \Lambda)\mathbf{u}_\mu+f(g, \Lambda)\partial_\mu\Lambda,\\ &\exp[ig(\boldsymbol{\tau}\cdot\phi)/2\kappa] \to \exp[ig(\boldsymbol{\tau}\cdot\phi')/2\kappa] = \exp[ig(\boldsymbol{\tau}\cdot\phi)/2\kappa]\exp[-ig(\boldsymbol{\tau}\cdot\Lambda)/2],\\ &f(-g, \phi/\kappa)\partial_\mu\phi \to f(-g, \phi'/\kappa)\partial_\mu\phi' = S(g, \Lambda)f(-g, \phi/\kappa)\partial_\mu\phi-\kappa f(g, \Lambda)\partial_\mu\Lambda,\end{aligned} \tag{6.9}$$

which is a point transformation satisfying our condition. In this case we

† The original expression for eq. (6.8) given by Gürsey is

$$-\tfrac{1}{2}\kappa^2[\mathbf{u}_\mu-ig^{-1}\,\mathrm{Tr}\,\boldsymbol{\tau}\,e^{-ig(\boldsymbol{\tau}\cdot\phi)/2\kappa}\,\partial_\mu\,e^{ig(\boldsymbol{\tau}\cdot\phi)/2\kappa}]^2+\ldots$$

have $\mathscr{L}_2 = 0$. Thus, one gets the following equivalent Lagrangian:

$$\mathscr{L}_G = \mathscr{L}_{\text{Y.M.}\,0}(\mathbf{A}_\mu, \mathbf{B}) + \mathscr{L}_{\text{Y.M. int}}(\mathbf{A}_\mu) + \tfrac{1}{2}\kappa^2 \mathbf{U}_\mu \cdot \mathbf{U}_\mu - \tfrac{1}{2}\kappa^2 \left[\mathbf{A}_\mu + \frac{1}{\kappa} f(-g, \boldsymbol{\phi}|k)\partial_\mu \boldsymbol{\phi}\right]^2. \quad (6.8')$$

6.3. GELL-MANN-LÉVY MODEL

For the Gell-Mann-Lévy model (in interaction with the Yang–Mills field[12]) we have

$$\begin{aligned}\mathscr{L}_{\text{G.L.}} = \mathscr{L}_{\text{Y.M.}} &- \bar{\psi}\{\gamma_\mu[\partial_\mu - \tfrac{1}{4}ig(1+\gamma_5)(\boldsymbol{\tau}\cdot\mathbf{u}_\mu)] - h[\sigma + i(\boldsymbol{\tau}\cdot\boldsymbol{\pi})\gamma_5]\}\psi \\ &- \tfrac{1}{2}[\partial_\mu \boldsymbol{\pi} - \tfrac{1}{2}g\boldsymbol{\pi}\wedge\mathbf{u}_\mu + \tfrac{1}{2}g\sigma\mathbf{u}_\mu]^2 - \tfrac{1}{2}[\partial_\mu\sigma - \tfrac{1}{2}g\boldsymbol{\pi}\cdot\mathbf{u}_\mu]^2 \quad (6.10) \\ &- \tfrac{1}{2}\mu^2[\boldsymbol{\pi}^2+\sigma^2] - \lambda\left[\boldsymbol{\pi}^2+\sigma^2 - \frac{1}{4f}\right]^2 - \frac{\mu^2}{2f}\sigma.\end{aligned}$$

The above Lagrangian is invariant, except for the last term and the mass term of $\mathscr{L}_{\text{Y.M.}}$, under the following transformation:

$$\begin{aligned}\mathbf{u}_\mu \to \mathbf{u}'_\mu &= S(g, \boldsymbol{\Lambda})\mathbf{u}_\mu + f(g, \boldsymbol{\Lambda})\partial_\mu \boldsymbol{\Lambda}, \\ \psi \to \psi' &= \exp[\tfrac{1}{4}ig(1+\gamma_5)(\boldsymbol{\tau}\cdot\boldsymbol{\Lambda})]\psi = \{\tfrac{1}{2}(1+\gamma_5)\exp[\tfrac{1}{2}ig(\boldsymbol{\tau}\cdot\boldsymbol{\Lambda})] + \tfrac{1}{2}(1-\gamma_5)\}\psi, \\ \sigma \to \sigma' &= \cos x\sigma + \frac{g}{2x}\sin x(\boldsymbol{\pi}\cdot\boldsymbol{\Lambda}), \quad (6.11) \\ \boldsymbol{\pi} \to \boldsymbol{\pi}' &= -\frac{g}{2x}\sin x\sigma\boldsymbol{\Lambda} + \cos x\boldsymbol{\pi} - \frac{g}{2x}\sin x(\boldsymbol{\Lambda}\wedge\boldsymbol{\pi}),\end{aligned}$$

where $x = \tfrac{1}{2}g\sqrt{(\boldsymbol{\Lambda}\cdot\mathbf{A})}$. Notice here that the last three transformations all satisfy our condition. In terms of x the transformation for $\mathbf{u}_\mu$ can be written as follows:

$$\begin{aligned}S(g, \boldsymbol{\Lambda})\mathbf{u}_\mu &= \frac{g}{x}\sin x\cos x(\mathbf{u}_\mu\wedge\boldsymbol{\Lambda}) + \frac{g^2}{2x^2}\sin^2 x\boldsymbol{\Lambda}(\boldsymbol{\Lambda}\cdot\mathbf{u}_\mu) + (1-2\sin^2 x)\mathbf{u}_\mu, \\ f(g, \boldsymbol{\Lambda})\partial_\mu\boldsymbol{\Lambda} &= \frac{g}{2x^2}\sin^2 x(\partial_\mu\boldsymbol{\Lambda}\wedge\boldsymbol{\Lambda}) - \frac{g^2}{4x^2}\left(\frac{1}{x}\sin x\cos x - 1\right)\boldsymbol{\Lambda}(\boldsymbol{\Lambda}\cdot\partial_\mu\boldsymbol{\Lambda}) \quad (6.12) \\ &\quad + \frac{1}{x}\sin x\cos x\partial_\mu\boldsymbol{\Lambda}.\end{aligned}$$

Thus, eq. (6.10) is equivalent to the following Lagrangian (using eq. (6.6)):

$$\begin{aligned}\mathscr{L}_{\text{G.L.}} = \mathscr{L}'_{\text{Y.M.}} &- \bar{\psi}\{\gamma_\mu[\partial_\mu - \tfrac{1}{4}ig(1+\gamma_5)(\boldsymbol{\tau}\cdot\mathbf{A}_\mu)] - h[\sigma + i(\boldsymbol{\tau}\cdot\boldsymbol{\pi})\gamma_5]\}\psi \\ &- \tfrac{1}{2}(\partial_\mu\boldsymbol{\pi} - \tfrac{1}{2}g\boldsymbol{\pi}\wedge\mathbf{A}_\mu + \tfrac{1}{2}g\sigma\mathbf{A}_\mu)^2 - \tfrac{1}{2}(\partial_\mu\sigma - \tfrac{1}{2}g\boldsymbol{\pi}\cdot\mathbf{A}_\mu)^2 \quad (6.10') \\ &- \tfrac{1}{2}\mu^2(\boldsymbol{\pi}^2+\sigma^2) - \lambda[\boldsymbol{\pi}^2+\sigma^2 - \tfrac{1}{4}f]^2 - \frac{\mu^2}{2f}\sigma',\end{aligned}$$

where $\sigma' = \cos x\,\sigma + (g/2x)\sin x\,(\boldsymbol{\pi}\cdot\mathbf{b})$, with $x = \tfrac{1}{2}g\sqrt{\mathbf{b}\cdot\mathbf{b}}$.

Appendix

A.1. PROOF OF THEOREM I

Let $\phi_r = \phi_r(\phi'_s)$ be a point transformation. Hence we have

$$\phi_r = \sum_{n=0}^{\infty} c_n \phi_s'^n, \tag{A.1}$$

where c_n may contain $\phi'_t (t \neq s)$. Hence we get

$$\dot{\phi}_r = \sum_{n=0}^{\infty} \{c_n (\underbrace{\dot{\phi}'_s \phi_s'^{n-1} + \phi'_s \dot{\phi}'_s \phi_s'^{n-2} + \ldots + \phi_s'^{n-1} \dot{\phi}'_s}_{n \text{ terms}}) + \dot{c}_n \phi_s'^n\}. \tag{A.2}$$

Hence by definitions (3.9)—(3.12) one has

$$\begin{aligned} \frac{\partial}{\partial \dot{\phi}'_s} \dot{\phi}_r &= \sum_{n=0}^{\infty} c_n (\phi_s'^{n-1} + \phi_s'^{n-1} + \ldots + \phi_s'^{n-1}) = \sum_{n=0}^{\infty} c_n n \phi_s'^{n-1} \\ &= \frac{\partial}{\partial \phi'_s} \sum_{n=0}^{\infty} c_n \phi_s'^n = \frac{\partial}{\partial \phi'_s} \phi_r \end{aligned} \tag{A.3}$$

as required.

A.2. PROOF OF THEOREM II

From eq. (A.2) and by using eqs. (3.9)—(3.12) it is easy to see that

$$\begin{aligned} \frac{\partial}{\partial \phi'_s} \dot{\phi}_r &= \sum_{n=0}^{\infty} [c_n \{\dot{\phi}'_s (n-1) \phi_s'^{n-2} + \phi'_s \dot{\phi}'_s (n-2) \phi_s'^{n-3} + \ldots + \phi'_s{}^{n-2} \dot{\phi}'_s \\ &\quad + \dot{\phi}'_s \phi'_s{}^{n-2} + \ldots + (n-2) \phi'_s{}^{n-3} \dot{\phi}'_s \phi'_s + (n-1) \phi'_s{}^{n-2} \dot{\phi}'_s\} + \dot{c}_n n \phi'_s{}^{n-1}] \\ &= \sum_{n=0}^{\infty} n c_n \{\dot{\phi}'_s \phi'_s{}^{n-2} + \phi'_s \dot{\phi}'_s \phi'_s{}^{n-3} + \ldots + \phi'_s{}^{n-3} \dot{\phi}'_s \phi'_s + \phi'_s{}^{n-2} \dot{\phi}'_s\} + \dot{c}_n n \phi'_s{}^{n-1}] \\ &= \sum_{n=0}^{\infty} \left[n c_n \frac{\partial}{\partial t} \{\phi'_s{}^{n-1}\} + \dot{c}_n n \phi'_s{}^{n-1} \right] \\ &= \frac{\partial}{\partial t} \sum_{u=0}^{\infty} c_n \frac{\partial}{\partial \phi'_s} \phi'_s{}^n = \frac{\partial}{\partial t} \frac{\partial}{\partial \phi'_s} \phi_r \end{aligned} \tag{A.4}$$

as required.

A.3. THE ENERGY-MOMENTUM TENSOR $T_{\mu\nu}$

Here we should like to prove $T_{\mu\nu} = T'_{\mu\nu}$ (eq. (3.16)). For a standard Lagrangian it is usual to define $T_{\mu\nu}$ as

$$T_{\mu\nu} = \sum_r \frac{1}{2} \left(\phi_{r,\mu} \frac{\partial \mathscr{L}}{\partial \phi_{r,\nu}} + \frac{\partial \mathscr{L}}{\partial \phi_{r,\nu}} \phi_{r,\mu} \right) - \delta_{\mu\nu} \mathscr{L}. \tag{A.5}$$

For a more general Lagrangian $\mathscr{L}$, however, this is inadequate, since, for

example, if $\mathscr{L}$ contains terms such as $f(\phi_r)\dot{\phi}_r^2$, the energy given by (A.5) is not conserved in general. To define $T_{\mu\nu}$ properly we first introduce a new operator [W $\partial/\partial Z$] which is defined analogously to $\partial/\partial Z$ (W, $Z = \phi_1$ or $\phi_2, \ldots$ or $\dot{\phi}_n$), by the following steps:

Step 1:

$$\left[W\frac{\partial}{\partial\phi_s}\right]\phi_r = \left[W\frac{\partial}{\partial\dot{\phi}_s}\right]\dot{\phi}_r = W\delta_{rs}, \qquad \left[W\frac{\partial}{\partial\dot{\phi}_r}\right]\phi_r = \left[W\frac{\partial}{\partial\phi_s}\right]\dot{\phi}_r = 0. \tag{A.6}$$

Step 2:

$$\left[W\frac{\partial}{\partial Z}\right](XY) = \left(\left[W\frac{\partial}{\partial Z}\right]X\right)Y + X\left(\left[W\frac{\partial}{\partial Z}\right]Y\right). \tag{A.7}$$

Note that this is not equal to $W\,\partial(XY)/\partial Z = W(\partial X/\partial Z)Y + WX\,\partial Y/\partial Z$, the difference being $[X, W]\partial Y/\partial Z$. The generalization of this is

$$\left[W\frac{\partial}{\partial Z}\right]\prod_{i=1}^{N}X_i = \sum_{j=1}^{N}\prod_{i=1}^{i-1}X_i\left(\left[W\frac{\partial}{\partial Z}\right]X_j\right)\prod_{i=j+1}^{N}X_i. \tag{A.8}$$

Step 3:

If F_i is a simple product as in Step 2 and $F = \sum_{i=1}^{M} F_i$, we put

$$\left[W\frac{\partial}{\partial Z}\right]F = \sum_{i=1}^{M}\left[W\frac{\partial}{\partial Z}\right]F_i. \tag{A.9}$$

The vital step is, of course Step 2. We can now define $T_{\mu\nu}$ in general. In fact,

$$T_{\mu\nu} \equiv \sum_r \left[\phi_{r,\mu}\frac{\partial}{\partial\phi_{r,\nu}}\right]\mathscr{L} - \delta_{\mu\nu}\mathscr{L}. \tag{A.10}$$

It is easily seen that this reduces to the usual definition when $\mathscr{L}$ contains the $\phi_{r,\mu}^2$ only with numerical coefficients.

Let us now prove the invariance of T_{44} under a point transformation (A.1). First we not that the standard $\mathscr{L}$ can be split into three parts,

$$\mathscr{L} = \mathscr{L}_{(2)} + \mathscr{L}_{(1)} + \mathscr{L}_{(0)}, \tag{A.11}$$

where $\mathscr{L}_{(2)}$, $\mathscr{L}_{(1)}$, $\mathscr{L}_{(0)}$ are bilinear, linear and of zero order, respectively, in the $\phi_{r,4}$. After making the point transformation we have, similarly,

$$\mathscr{L}'(=\mathscr{L}) = \mathscr{L}'_{(2)} + \mathscr{L}'_{(1)} + \mathscr{L}'_{(0)}, \tag{A.11'}$$

where $\mathscr{L}'_{(2)}$, $\mathscr{L}'_{(1)}$, $\mathscr{L}'_{(0)}$ are bilinear, linear, and of zero order, respectively, in the $\phi'_{r,4}$. The point nature of the transformation guarantees, however, that not only is $\mathscr{L} = \mathscr{L}'$, but also

$$\mathscr{L}_{(2)} = \mathscr{L}'_{(2)}, \qquad \mathscr{L}_{(1)} = \mathscr{L}'_{(1)}, \qquad \mathscr{L}'_{(0)} = \mathscr{L}'_{(0)}. \tag{A.12}$$

Next we note that by its definition (A.6)–(A.9), the operator $\sum_r[\phi_{r,4}\,\partial/\partial\phi_{r,4}]$ has the property that acting on a linear function of the $\phi_{r,4}$(e.g. $\phi^2\phi_{,4}+\phi_{,4}\phi^2$) it multiplies it by unity, and acting on a quadratic function (e.g $\phi\phi_{,4}{}^2+\phi_{,4}{}^2\phi$) it multiplies it by a factor 2. Hence we have

$$T_{44}=2\mathscr{L}_{(2)}+\mathscr{L}_{(1)}-\mathscr{L}, \qquad T'_{44}=2\mathscr{L}'_{(2)}+\mathscr{L}'_{(1)}-\mathscr{L}', \tag{A.13}$$

and therefore, by (A.12) one has

$$T_{44}=T'_{44}, \tag{A.14}$$

as required.

Similar proofs obtain for $T_{i4}=T'_{i4}$ and $T_{ij}=T'_{ij}$.

A.4. POINT TRANSFORMATIONS FOR FERMION VARIABLES

On account of the *anti*-commutation properties of fermion variables some new definitions of derivatives, etc. are needed in this case. We deal first with these new definitions.

Let X_r be a set of n anti-commuting quantities $\{X_s, X_r\}=0, r, s=1, 2, \ldots, n$. We consider variable transformations $X'_r(X_s)$ of the X_r such that the X'_s can be expressed as sums of products of the X_r, with an *odd* number of X_r in each product, and conversely †. As a result one has

$$\{X_r, X_s\}=\{X_r, X'_s\}=\{X'_r, X'_s\}=0. \tag{A.15}$$

(Note that since $X_r{}^2=0$, each product can contain any given X_r at most linearly.) We next define drivatives of any functions F of the X_r, $\dot{X}_r$ and $X_r{}^\dagger$ which can be expressed as sums of products (not necessarily odd here) of these quantities ($X_r{}^\dagger$ means the Hermitian conjugate of X_r and is taken to be variationally independent of X_r) as follows:

Step 1:

$$\frac{\partial X_s}{\partial X_r}=\frac{\partial \dot{X}_s}{\partial \dot{X}_r}=\frac{\partial X_s{}^\dagger}{\partial X_r{}^\dagger}=\delta_{rs}, \qquad \frac{\partial X_r}{\partial X_s{}^\dagger}=\frac{\partial \dot{X}_r}{\partial X_s}=\frac{\partial \dot{X}_r{}^\dagger}{\partial \dot{X}_s}=0, \quad \text{etc.} \tag{A.16}$$

Step 2:

With $X, Y, Z, \ldots U=X_1, \ldots, X_r \ldots, \dot{X}_r, \ldots, X_r{}^\dagger, \ldots, \dot{X}_n{}^\dagger$ we put

$$\frac{\partial}{\partial U}(XYZ\ldots)=\frac{\partial X}{\partial U}YZ-X\frac{\partial Y}{\partial U}Z\ldots+\ldots \tag{A.17}$$

(note the minus sign: a plus sign would lead to the contradiction

$$2X_qX_p=\frac{\partial X_q}{\partial X_p}X_p{}^2+X_q\frac{\partial X_p}{\partial X_p}X_p+X_qX_p\frac{\partial X_p}{\partial X_p}=\frac{\partial}{\partial X_p}(X_qX_p{}^2)=\frac{\partial 0}{\partial X_p}=0\Big).$$

† An example of the sort of function we consider is $X'=X+XYZ$, $Y'=Y+YZX$, $Z'=Z+ZXY$; $X=X'-X'Y'Z'$, $Y=Y'-Y'Z'X'$, $Z=Z'-Z'X'Y'$

Step 3:
If the F_i are simple products as in Step 2, we put

$$\frac{\partial}{\partial X_s}\sum_i F_i = \sum_i \frac{\partial F_i}{\partial X_s}. \tag{A.18}$$

With these definitions we establish the following three theorems :
Theorem I':

$$\frac{\partial F(X'_s(X_r))}{\partial X_r} = \sum_s \frac{\partial X'_s}{\partial X_r}\frac{\partial F(X'_s)}{\partial X'_s}, \tag{A.19}$$

where X'_s is a sum of odd products of X_r.
Theorem II':

$$\frac{\partial}{\partial t}\frac{\partial}{\partial X_r}F(X_s) = \frac{\partial}{\partial X_r}\frac{\partial}{\partial t}F(X_s). \tag{A.20}$$

Theorem III':

$$\frac{\partial}{\partial \dot{X}_r}\frac{\partial}{\partial t}F(X_s) = \frac{\partial}{\partial X_r}F(X_s). \tag{A.21}$$

The last two are the analogues of theorems I and II of the boson case. The first is not as trivial as it looks and is not true unless the X'_s are sums of *odd* products of the X_r. All these theorems can be proved by induction and as the proof is rather simple, we shall not give it explicitly.

We are now in a position to consider the invariance of the Heisenberg-Pauli formalism under point transformations. Suppose we have a standard Lagrangian density

$$\mathscr{L} = -\tfrac{1}{2}\bar{\psi}(\gamma\overrightarrow{\partial}+\kappa)\psi+\tfrac{1}{2}\bar{\psi}(\gamma\overleftarrow{\partial}-\kappa)\psi+\bar{\psi}O\psi \tag{A.22}$$

with the usual notation, and where O is a function of the γ-matrices and some boson field only. The Heisenberg-Pauli equations $(\partial/\partial t)\,(\partial\mathscr{L}/\partial\dot{\phi}) = \delta\mathscr{L}/\delta\phi$ and $\{\partial\mathscr{L}/\partial\dot{\phi}\,\phi\} = i\Delta$ (Δ is a c-number) are in this case, according to our definitions (A.16)–(A.18), and regarding $\bar{\psi}$ and ψ as variationally independent, with $\partial\mathscr{L}/\partial\dot{\psi}_\alpha = -\frac{1}{2}i(\bar{\psi}\gamma_4)_\alpha$, $\partial\mathscr{L}/\partial\dot{\bar{\psi}}_\alpha = -\frac{1}{2}i(\gamma_4\psi)_\alpha$,

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{1}{2i}(\bar{\psi}\gamma_4)_\alpha = \kappa\psi_\alpha-(\bar{\psi}\gamma_i\overleftarrow{\partial}_i)_\alpha-(\bar{\psi}0)_\alpha-\frac{1}{2i}(\dot{\bar{\psi}}\gamma_4)_\alpha, \qquad \left\{\frac{1}{2i}(\bar{\psi}\gamma_4)_\alpha, \psi_\beta\right\} = i\Delta\delta_{\alpha\beta},\\
&\frac{\partial}{\partial t}\frac{1}{2i}(\gamma_4\psi)_\alpha = -\kappa\psi_\alpha-(\gamma_i\overrightarrow{\partial}_i\psi)_\alpha+(0\psi)_\alpha-\frac{1}{2i}(\gamma_4\dot{\psi})_\alpha, \quad \left\{\frac{1}{2i}(\gamma_4\psi)_\alpha, \bar{\psi}_\beta\right\} = i\Delta\delta_{\alpha\beta},
\end{aligned} \tag{A.23}$$

which are the usual equations for a spinor field. We now make a point transformation, which we define to be

$$\psi_\alpha = \psi'_\alpha F(\psi'_\beta), \tag{A.24}$$

where F is a function only of bilinear forms of ψ'. [F does not contain $\dot{\psi}'$,

just as in the boson case, but in addition F does not contain $\bar{\psi}'$, which is a restriction peculiar to the fermion case and is imposed because $\{\psi, \bar{\psi}\} \neq 0$. Covariant bilinear forms of ψ' alone can be constructed by means of the B-matrix of Pauli [13]); for example: $\bar{\psi}B\gamma_\mu\chi$ is a vector.] The transformation (A.24) is automatically of the type stipulated above (assuming, of course, that F can be expressed as a power series in its argument) and theorems I′, II′ and III′ apply.

We now prove the invariance of the Heisenberg-Pauli formalism (A.23) under the transformation (A.24). From (A.22) we get, using theorem III′,

$$\frac{\partial \mathscr{L}}{\partial \dot{\psi}'_\alpha} = \frac{1}{2i}(\bar{\psi}\gamma_4)_\beta \frac{\partial \dot{\psi}_\beta}{\partial \dot{\psi}'_\alpha} + \frac{1}{2i}\frac{\partial \dot{\bar{\psi}}_\beta}{\partial \dot{\psi}'_\alpha}(\gamma_\alpha \psi)_\beta = \frac{1}{2i}(\bar{\psi}\gamma_4)_\beta \frac{\partial \dot{\psi}_\beta}{\partial \dot{\psi}'_\alpha} = \frac{1}{2i}(\bar{\psi}\gamma_4)_\beta \frac{\partial \psi_\beta}{\partial \psi'_\alpha}.$$

Thus, one has, according to eq. (A.23), theorem II′ and a slight extension of theorem I′ to cover functional derivatives,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{d\dot{\psi}'_\alpha} &= \left[\frac{\partial}{\partial t}\frac{1}{2i}(\bar{\psi}\gamma_4)_\beta\right]\frac{\partial \psi_\beta}{\partial \psi'_\alpha} + \frac{1}{2i}(\bar{\psi}\gamma_4)_\beta \frac{\partial}{\partial t}\left(\frac{\partial \psi_\beta}{\partial \psi'_\alpha}\right) \\ &= \frac{\delta \mathscr{L}}{\delta \psi_\beta}\frac{\partial \psi_\beta}{\partial \psi'_\alpha} + \frac{1}{2i}(\bar{\psi}\gamma_4)_\beta \frac{\partial}{\partial t}\left(\frac{\partial \psi_\beta}{\partial \psi'_\alpha}\right) = \frac{\delta \mathscr{L}}{\delta \psi_\beta}\frac{\partial \psi_\beta}{\partial \psi'_\alpha} + \frac{1}{2i}(\bar{\psi}\gamma_4)_\beta \frac{\partial \dot{\psi}_\beta}{\partial \psi'_\alpha} \\ &= \frac{\delta \mathscr{L}}{\delta \psi_\beta}\frac{\partial \psi_\beta}{\partial \psi'_\alpha} + \frac{\partial}{\partial \psi'_\alpha}\left(\frac{-1}{2i}\bar{\psi}\gamma_4\dot{\psi}\right) = \frac{\delta \mathscr{L}}{\delta \psi'_\alpha}.\end{aligned}$$

On the other hand, since $[\partial\psi_\gamma/\partial\psi'_\alpha, \psi'_\beta] = 0$, and using theorem I′, one has

$$\begin{aligned}\left\{\frac{\partial \mathscr{L}}{\partial \dot{\psi}'_\alpha}, \psi'_\beta\right\} &= \left\{\frac{1}{2i}(\bar{\psi}\gamma_4)_\gamma \frac{\partial \psi_\gamma}{\partial \psi'_\alpha}, \psi'_\beta\right\} = \frac{1}{2i}\{(\bar{\psi}\gamma_4)_\gamma, \psi'_\beta\}\frac{\partial \psi_\gamma}{\partial \psi'_\alpha} \\ &= \frac{\partial \psi'_\beta}{\partial \psi_\gamma}\frac{\partial \psi_\gamma}{\partial \psi'_\alpha} i\Delta = \frac{\partial \psi'_\beta}{\partial \psi'_\alpha} i\Delta = \delta_{\beta\alpha} i\Delta.\end{aligned}$$

Similar equations obtain for $\partial\mathscr{L}/\partial\dot{\bar{\psi}}'_\alpha$. The invariance of $T_{\mu\nu}$ can be established here in the same way as in the boson case. Thus, the invariance of the Heisenberg-Pauli formalism is established.

Here, we have considered only one fermion field, but it is very probable that the above considerations can be extended to the case of several fermion fields, possibly interacting directly (as in the case of universal Fermi interactions).

The last question we must consider is a point transformation which mixes fermion and boson variables. For simplicity, and because it is only cases of this kind which we have had to consider in applications, we take a Lagrangian of the type

$$\mathscr{L} = -\tfrac{1}{2}\bar{\psi}(\gamma\overrightarrow{\partial} + \kappa)\psi + \tfrac{1}{2}\bar{\psi}(\gamma\overleftarrow{\partial} - \kappa)\psi + g\bar{\psi}0\psi + \tfrac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2 + \mu^2\phi^2). \quad \text{(A.25)}$$

It is then easy to establish along the lines of section 3 and of this Appendix that the Heisenberg-Pauli formalism for this Lagrangian is invariant under the point transformation $\psi_\alpha = \psi'_\alpha F(\psi', \phi')$, $\phi = f(\phi')$, where F is a function of ϕ' and of bilinear forms of ψ' and F and f are, as usual, expressible as (possibly infinite) power series of their arguments.

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EXCHANGE STRIPPING EFFECTS IN THE REACTION $B^{11}(d, n)C^{12}_{4.43}$

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Abstract: Angular distributions for the $B^{11}(d, n)\ C^{12}_{4.43}$ reaction are presented for bombarding energies of 1.1, 2.5 and 2.8 MeV (Lab.). These distributions are fitted with a two-mode stripping theory. This theory with all parameters fixed is used to fit n–γ correlations at 1.1 MeV and published data at 2.65 MeV. It is suggested that the descrepancies which still remain between theory and experiment may be reconciled by using a distorted wave Born approximation to the two-mode theory.

1. Introduction

Garg, Gale and Calvert [1]) have recently investigated n–γ correlations of the neutron group leading to the 4.43 MeV level in C^{12} and the associated γ rays from the reaction $B^{11}(d, n)C^{12}$ in an effort to determine the contribution of the heavy particle stripping mode [2, 3]) to the reaction mechanism. The purpose of this note is to report preliminary results concerning the same correlations at a lower deuteron bombarding energy and to compare the predictions of the theory with our data and to that of GGC. The agreement is sufficiently close to suggest that their conclusions should be modified.

2. Experimental Arrangement

The neutron angular distribution and the n-γ correlation has been measured utilizing a time of flight spectrometer previously described [4]). A NaI(Tl) crystal was used to detect the 4.43 MeV gamma ray from the decay of the first excited state of C^{12}. This pulse was used to start the timing period; a neutron pulse detected in a pilot B scintillator after a flight path of 100 cm was used to stop the timing period. The detectors were shielded from fast electrons by 0.6 cm of Pb. The gamma detector could be rotated in the plane defined by the deuteron beam and the neutron counter for angular correlation measurements. The gamma detector could also be placed at 90° to the deuteron-

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neutron plane in order to measure the neutron angular distributions. Fig. 1 describes our definitions of angles.

A typical time of flight spectrum is shown in fig. 2.

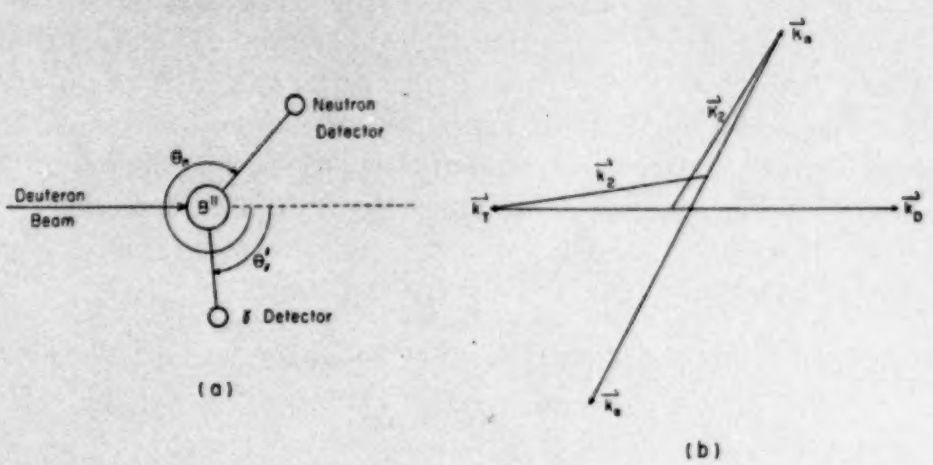


Fig. 1. (a) Angular definitions in the reaction plane. (b) A wave vector diagram for the B^{11}(d, n) C^{12} reaction.

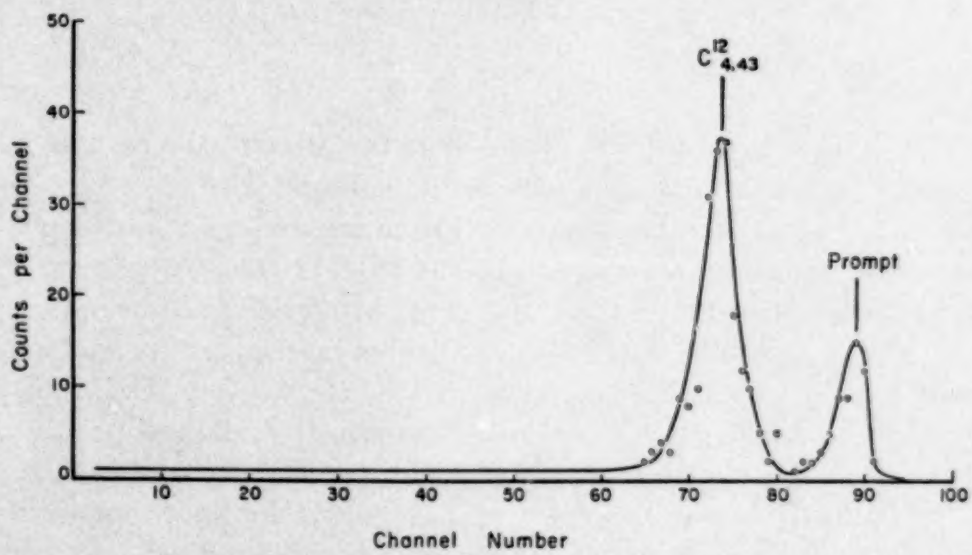


Fig. 2. A typical time of flight spectrum.

3. Theoretical Analysis

The theoretical analysis of n-γ correlations for the reaction B^{11}(d, n)C^{12} including the heavy particle stripping mode has been given earlier [2, 3]). In this analysis the stripping radii for the two modes and the ratio of their reduced widths are determined by fitting the experimental neutron angular distribution at the same energy and these parameters are then used to calculate the correlation function corresponding to the various neutron angles. Hence, once the distribution is fitted the correlations are determined and no further adjustment is possible.

Using the notation of ref. [3]) the following quantum numbers are used to calculate the angular distribution in the j-j coupling scheme: $j_D = 1$; $j_n = \frac{1}{2}$; $j_p = \frac{1}{2}$; $J_p = \frac{1}{2}$; $l_p = 1$; $J_e = 2$; $J_T = \frac{3}{2}$; $J_f = 0$; $L = 2$; $J_n = \frac{3}{2}$; $j_c = 3$; $l_n = 1$; $l_c = 0$; $J_c = 3$. It should be noted that in some cases the specification of these quantum numbers is not sufficient to determine the sign of the interference term. Since there is no a priori guidance for the order of the addition of the spins and since in some cases a reversal in the order of addition changes the phase of the Clebsch-Gordan coefficient the phase of the entire interference term is in doubt. The sign was determined by a best fit criterion.

In deriving the theoretical expression the angular correlation is obtained first. Using the quantum numbers above the expression is

$$W(\theta_n, \theta_\gamma) = G_D^2 f_D^2 + 0.211\, G_H^2 f_H^2 X^2 \{1 + 0.621 \cos^2 \theta_\gamma\} + 1.39\, G_D f_D G_H f_H \{1 + 0.428 \cos^2 \theta_\gamma\} X \cos \beta. \quad (1)$$

Then this correlation expression is integrated over all gamma angles to obtain the angular distribution

$$\frac{d\sigma(\theta_n)}{d\Omega} = G_D^2 f_D^2 + 0.254\, G_H^2 f_H^2 X^2 + 1.59\, G_D f_D G_H f_H X \cos \beta. \quad (2)$$

(Note that in eq. (1) the angle θ_γ is measured relative to the $\mathbf{K}_2$ axis. In the figures the θ'_γ shown is measured relative to the beam axis.)

It should be noted that the heavy particle term supplies a non-isotropic contribution to the correlation even though $l_c = 0$. This of course arises from the fact that the separated neutron and the core are not in a relative S state in the target nucleus. Also it should be noted that the deuteron stripping term supplies an isotropic contribution to the correlation even though $l_p = 1$. Hence, even though terms up to P_2 ($\cos \theta_\gamma$) are possible in principle, the theory warrants a detailed calculation to determine what terms are actually predicted. Indeed, even the symmetry axis cannot be predicted a priori in the two-mode theory without a detailed calculation, for in general one does not expect symmetry about either axis of quantization but about some intermediate axis.

The above expressions were calculated with the two-mode stripping theory in plane wave Born approximation. The heavy particle stripping mode should not be used alone in order to account for any data to the exclusion of the deuteron stripping mode. It is also incorrect to assume that the two modes add incoherently. Since the exchange mode introduces another term in the reaction amplitude [3]), they will be coherent in general. Only in cases of special symmetries will the interference term be zero. These two modes are different physically and should, in a theoretical analysis, complement one another; indeed quite frequently a relatively small fraction of heavy particle stripping need be added to account for the data.

Of course, the incoming particle cannot be adequately described by plane waves and a better approximation to the physical situation is to use distorted waves in the matrix elements of the two-mode theory. However, the latter analysis, which is our future goal, requires a computer and additional experimental data which are not presently at hand, hence the plane wave approximation will be used to fit the present data.

4. Results and Discussion

Figs. 3 to 5 show the angular distributions measured at 1.1, 2.5 and 2.8 MeV (Lab.), respectively. The solid curves are obtained from eq. (2) by using a Hulthén wave function for the deuteron in order to evaluate the $G_D(K_1)$. A square well potential is used to generate the wave functions used in $G_H(K_2)$, $f_D(k_1 R_1)$ and $f_H(k_2 R_2)$. The variable parameters have values as shown in table 1.

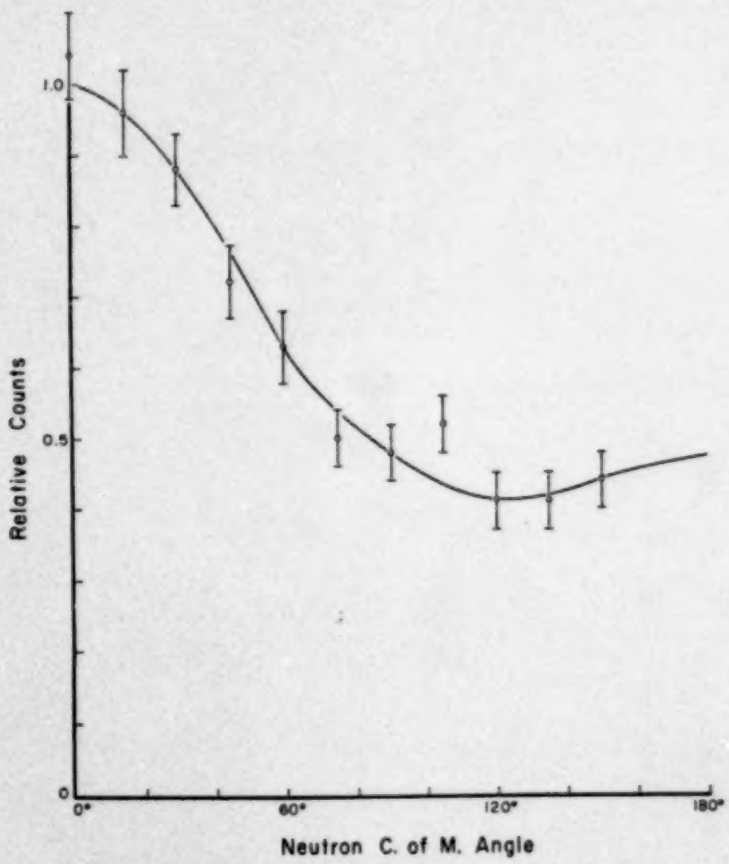


Fig. 3. Neutron angular distribution for B^{11} (d, n) $C^{12}_{4.43}$ at a bombarding energy of 1.1 MeV. The solid curve is calculated from eq. (2) with the parameters $R_1 = 5.5$ fm, $R_2 = 2.7$ fm, $X = 0.136$, $\theta_n = 345°$.

The correlations as given by eq. (1) are now fixed and the fits to our data at 1.1 MeV for a neutron angle of 345° is shown in fig. 6.

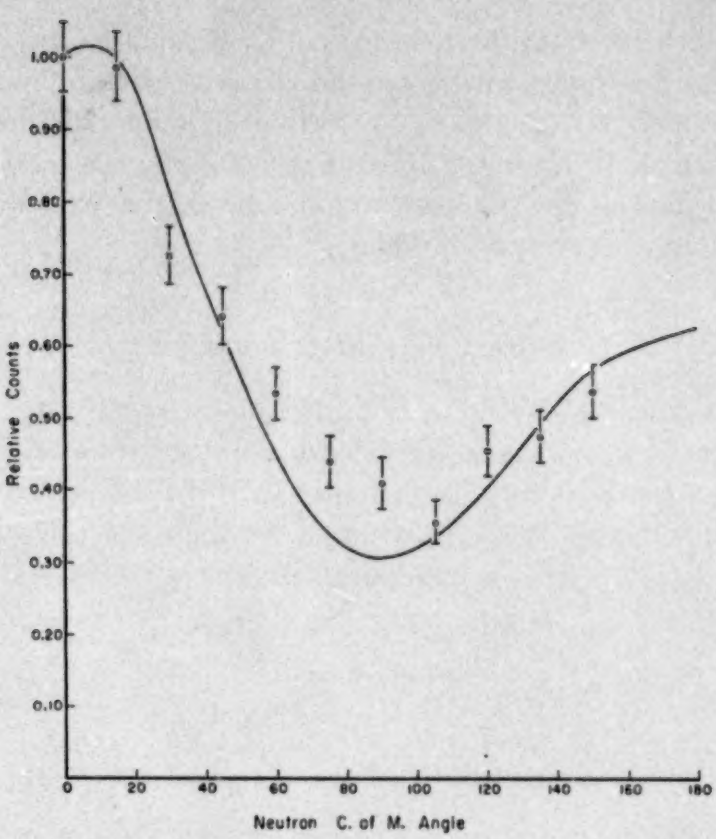


Fig. 4. Neutron angular distribution for B^{11} (d, n)$C^{12}_{4.43}$ at a bombarding energy of 2.5 MeV. The solid curve is calculated from eq. (2) with the parameters $R_1 = 5.5$ fm, $R_2 = 2.7$ fm, $X = 0.140$

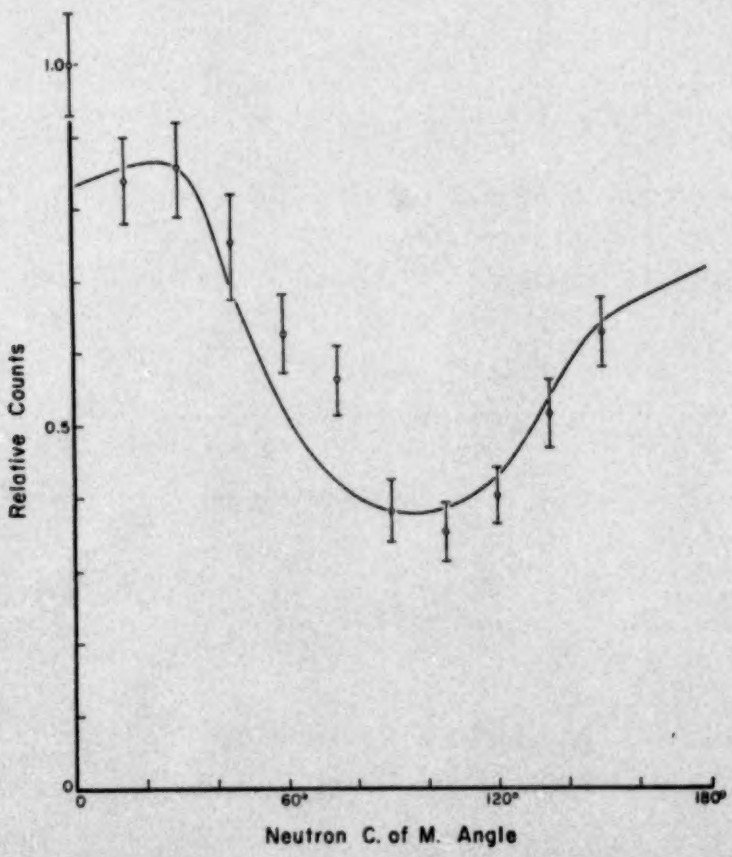


Fig. 5. Neutron angular distribution for B^{11} (d, n)$C^{12}_{4.43}$ at a bombarding energy of 2.8 MeV. The solid curve is calculated from eq. (2) with the parameters $R_1 = 4.55$ fm, $R_2 = 2.7$ fm, $X = 0.136$.

TABLE 1

Parameter values

	1.1 MeV	2.5 MeV	2.8 MeV
R_1 (fm)	5.5	5.5	4.55
R_2 (fm)	2.7	2.7	2.7
X	0.136	0.140	0.136

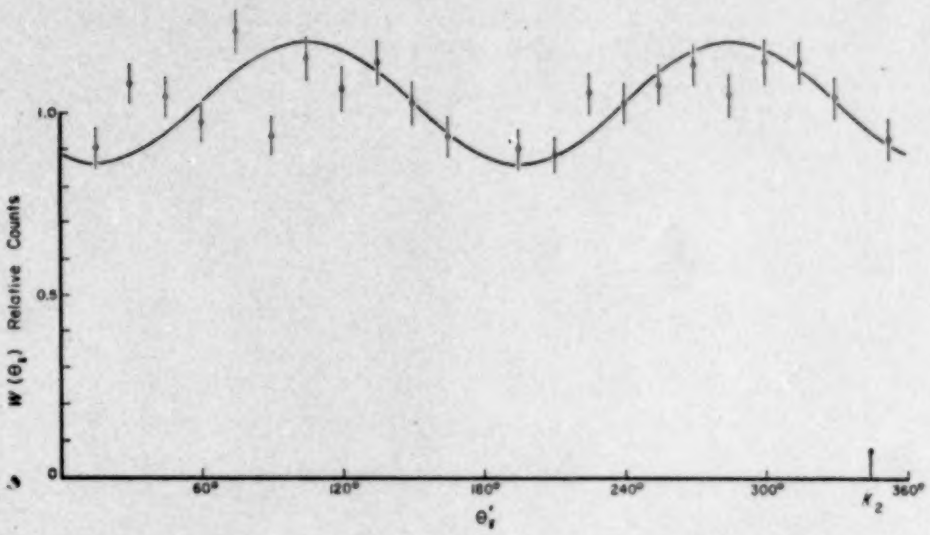


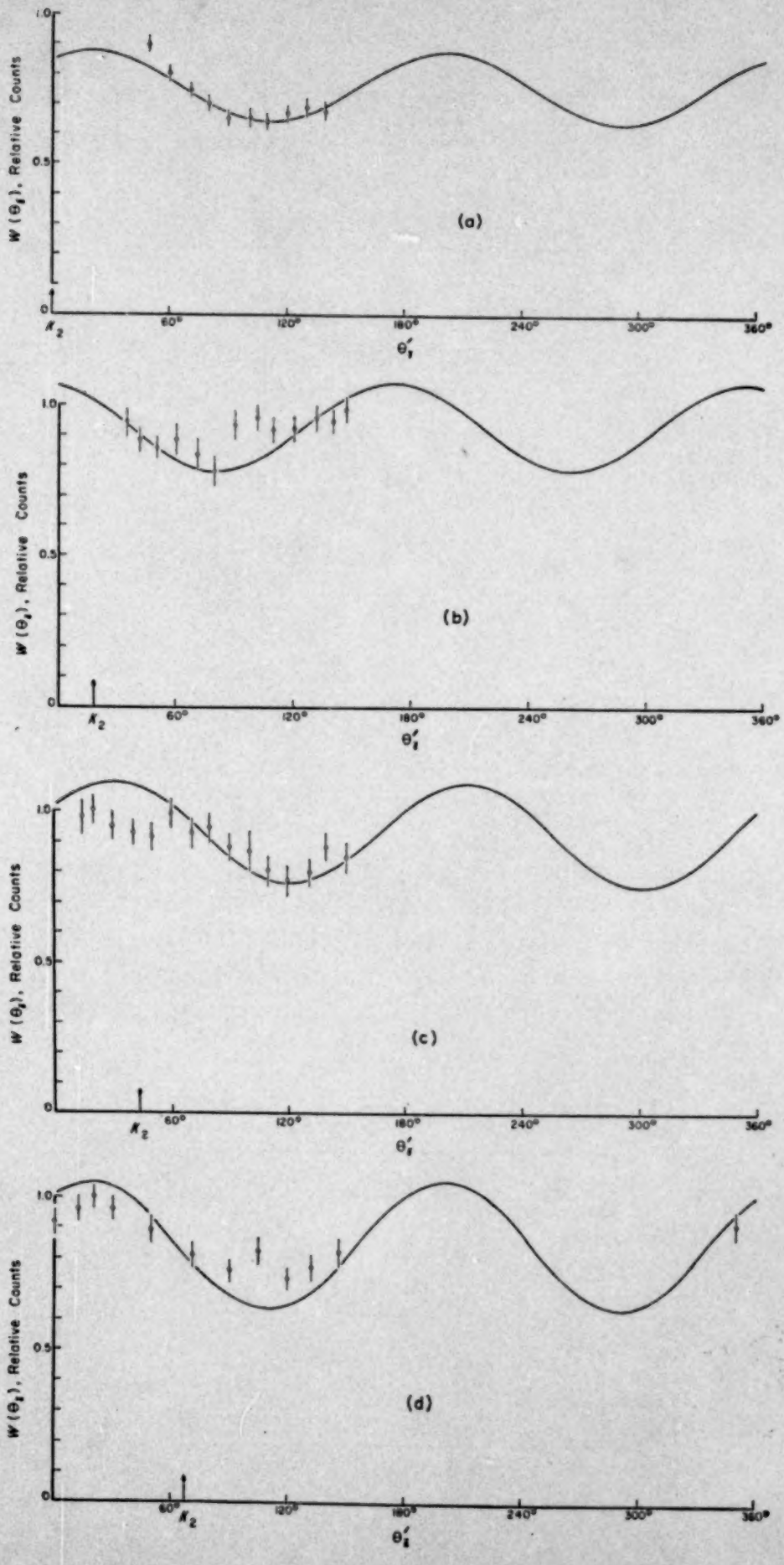
Fig. 6. A γ angular correlation for a bombarding energy of 1.1 MeV. The solid curve is calculated from eq. (1) using a shift in symmetry axis of 60°.

The correlations predicted for 2.5 MeV have been used to fit the data of ref. [1]) at 2.65 MeV (fig. 7). The calculated correlations at 15°, 75° and 105° have been used to fit the GGC data of 20°, 70° and 110°, respectively. A close inspection of the variation of the theoretical correlations as a function of angle and energy make us confident that these small extrapolations are justified.

TABLE 2

Shift in the symmetry axis

	θ_n	Shifts
1.1 MeV	345°	60°
2.5 MeV	0°	−20°
	20°	35°
	45°	15°
	70°	50°
	90°	40°
	110°	70°
	135°	110°


W (θ'1), Relative Counts
1.0
0.5
0
(a)
K2
60°
120°
180°
240°
300°
360°
θ'1
(b)
(c)
(d)

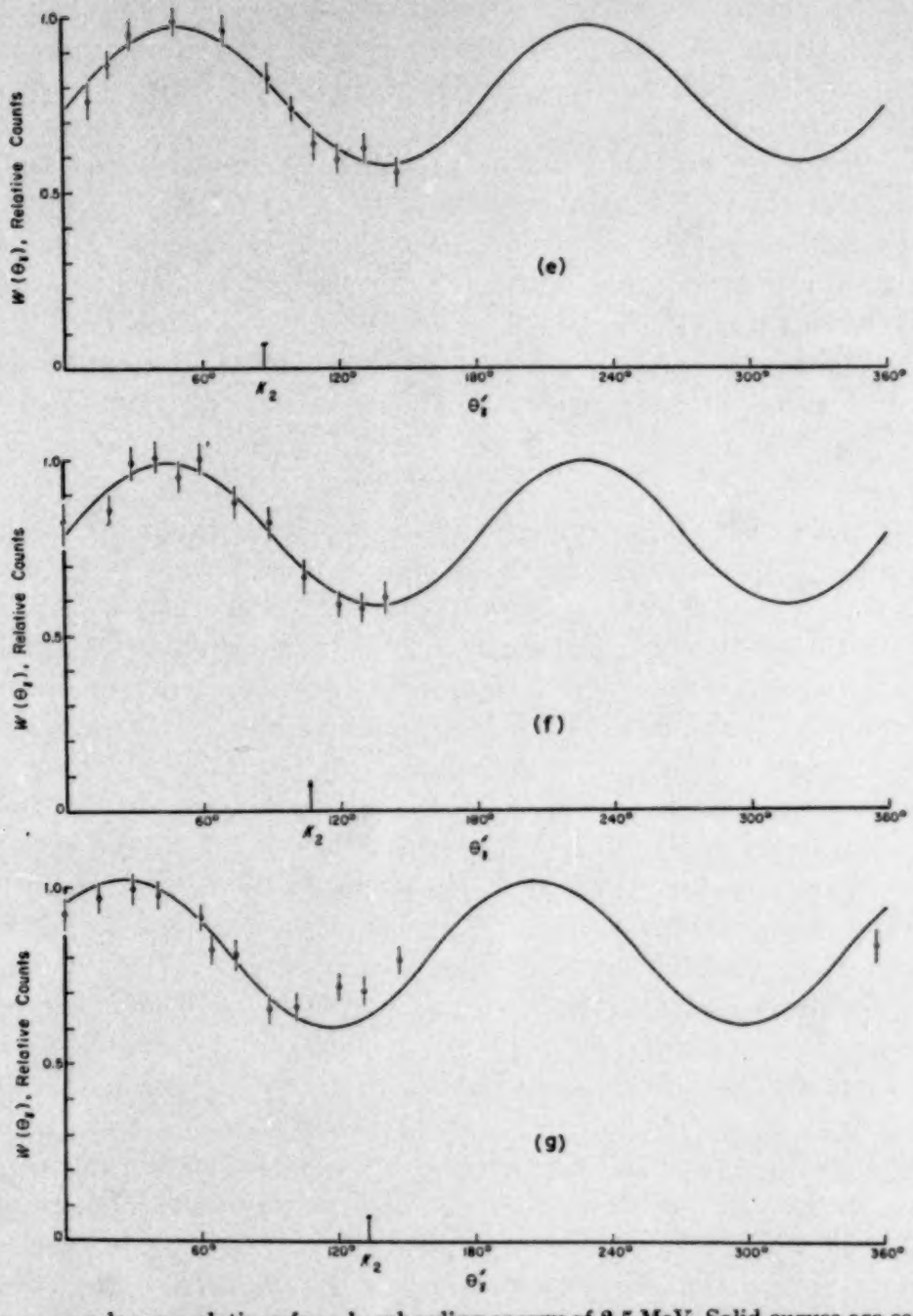


Fig. 7. n–γ angular correlations for a bombarding energy of 2.5 MeV. Solid curves are calculated from eq. (1) with shift in the symmetry axis as shown in table 2. The data are that of ref. [1]) at 2.65 MeV. a) $\theta_n = 0°$; b) $\theta_n = 20°$; c) $\theta_n = 45°$; d) $\theta_n = 70°$; e) $\theta_n = 90°$; f) $\theta_n = 110°$; g) $\theta_n = 135°$.

When the correlations were fitted to the experimental data the axis of symmetry predicted (which for this particular case corresponds to the $\mathbf{K_2}$ axis) was shifted in order to obtain the best agreement with experiment. This shift in the symmetry axis is assumed to be a manifestation of the effects of wave distortion on the stripping phenomenon. The shift in symmetry axis

for the correlations which have been fit is shown in table 2. It is hoped that a calculation of this two-mode reaction in distorted wave Born approximation might shed some light on the apparent trend of this shift to increase with backward neutron angle.

The value of the axis shift for the 2.65 MeV data is somewhat ambiguous because of an apparent contradiction in the GGC data. We have used for the purpose of this analysis the interpretation that θ_n defines the angle between the neutron counter and the deuteron beam (see fig. 1). Because of the high Q value of this reaction, $\mathbf{K}_2$ should be nearly in the direction of θ_n. We have neglected the fact that $\mathbf{K}_2$ is plotted on the GGC curves and assumed that their θ_n corresponds to ours, as is evident from their angular distributions.

5. Conclusions

The fits which have been presented for the angular correlations and distributions for the B^{11}(d, n)C^{12} indicate to us that a direct process including heavy particle stripping supplies the major contribution to this reaction. The presence of shifts in the symmetry axis of the correlation function indicate that the plane wave approximation is not sufficient for a complete understanding of the problem and that the distorted wave approximation of two-mode stripping should be pursued in a complete analysis.

One further comment should be added concerning the amplitude of the heavy particle term. Day *et al.* [5]) have recently noted that the exchange of heavy-particle amplitude is reduced approximately by a factor M_n/M_t (M_n is the neutron mass and M_t is the target mass), when distorted outgoing waves are taken into account. This factor is to be compared to a factor $\frac{1}{2}$ appearing with the deuteron stripping mode. Their analysis is based upon the problem of electron scattering from atomic hydrogen in which there is but one electron available for the exchange process. In nuclear stripping problems, the number of nucleons in the target nucleus available for the stripping mode must be included. Thus for the particular reaction considered here in which the four outer shell neutrons are incorporated in the amplitude, the heavy particle amplitude goes as $\frac{4}{11}$ as compared to $\frac{1}{2}$ for the deuteron mode. These slight variations are incorporated in the parameters A_j, and do not affect the arguments presented here.

Also, one might argue that the probability for capture of two nucleons on the surface of a B^{10}-like core might be small. If this is the case one should expect the cross sections for the reaction $B^{10}(He^3, p)C^{12}$ to be very small. These cross sections are in fact of the same order as those for deuteron stripping [6]) and exhibit the same characteristics of a direct interaction process. Therefore the presence of the heavy-particle mode must be taken into account.

We wish to thank Dr. G. Feldman for his stimulating discussion.

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CROSS-SECTIONS FOR (n, 2n) REACTIONS AT 14.8 MeV

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Abstract: Total cross-sections for (n, 2n) reactions have been measured at 14.8 MeV neutron energy for more than twenty cases by the activation method. Discrepancies have been found in some cases from the published values. Also a number of new cases have been measured. The observed values have been compared with those calculated on the statistical model using level density formulae of Lang and Le Couteur [9]) and Newton [10]). Comparison of the calculated and observed cross-sections gives an indication that at least part of the (n, 2n) reaction goes through some mechanism other than the formation of the compound nucleus.

1. Introduction

The (n, 2n) cross-sections have been measured earlier by many workers [1–5]). Paul and Clarke [1]) made an extensive study of these cross-sections, along with (n, p) and (n, α) cross-sections. They compared their values with those calculated with the statistical model using the level density formula suggested by Weisskopf [6]). The comparison showed that, in general, the measured values are explained by the statistical model; though, in a number of cases the measured values were less than the calculated ones. Recently, however, the measurements of the energy distribution of outgoing neutrons in (n, 2n) reactions indicated that the mechanism of interaction in medium weight and heavy nuclei is such that, while the emission of the second neutron proceeds through the compound nucleus, the first neutron is emitted through direct interaction[7]). The angular distribution of (n, 2n) neutrons also gives some indication of direct interaction [8]).

The present work was undertaken firstly to make consistent measurements of (n, 2n) cross-sections which included some of the previously measured cases with large quoted errors and some new cases; and secondly to test the validity of the compound nucleus model at these energies. The cross-sections have been calculated with the statistical compound nucleus model using the level density formulae of Lang and Le Couteur [9]) and Newton [10]).

2. Experimental

The experimental procedure adopted for irradiating the samples and measuring the activities produced in the samples has already been described in detail in an earlier paper [11]). All the irradiations were made in the forward direction and at the back of the tritium target, so as to avail oneself of the maximum

flux. Neutron yields of $\approx 10^9$ n/sec were obtained at the target. Because of the small energies of the incident deuterons (< 150 keV), and the fact that we made all irradiations in the forward direction with respect to the incident deuterons, the spread in the energy of the neutrons falling on the samples was not more than 0.5 MeV. The mean value of the energy of the incident neutrons may be taken as 14.8 MeV.

An end-window beta-counter of 4 cm diameter, with a mica window of 4 mg/cm^2 thickness, was used for measuring the activities of the irradiated samples. The length of the counter was kept only 3 cm to reduce the background, which was 15 counts/min when the counter was properly surrounded by lead.

3. Measurements

The cross-sections reported here have been measured by comparing them to the cross-section of the $Fe^{56}(n, p)Mn^{56}$ reaction, for which the value of 126 mb at 14.8 MeV was adopted. This value is based on the values of 96.7 mb $\pm$ 12% at 14.5 MeV measured by Paul and Clarke [1]), 124$\pm$12 mb at 14.1 MeV by Forbes [2]), 144$\pm$19 mb at 14.0 MeV by Yassumi [5]) and 131$\pm$15 mb at 15.27 MeV by B. D. Kern *et al.* [12]). The relative activation curve of Terrell and Holm [13]) was used to determine the corresponding value at 14.8 MeV for each of the above values.

Iron as a standard has many advantages over other elements. Firstly, its cross-section has been measured by many workers and their values are in agreement within small errors. Secondly, the cross-section being quite high, it gives a strong activity even for short irradiation times. Thirdly, it produces a pure activity of half life 150 min, which is suitable for comparison with other activities with a wide range of half lives.

The targets to be studied along with the iron samples were irradiated by neutrons for appropriate times which differed for the various cases. For long half lives of the order of days or several hours, the irradiation time was about 3 h. For cases of shorter half lives, the irradiation time was about half an hour. Precautions were taken to keep the neutron flux as constant as possible.

The activities of the irradiated targets were followed till the counting rate came down to that of the background. In this way it was assured that every activity was properly taken into account. For cases of low activities, the measurements were repeated several times to improve the statistics. When the decay schemes of the activities to be studied were complex, e.g. Cl^{34m}, Ga^{68}, Mo^{99} and Yb^{175}, the most recent decay schemes [17]) were used for the purpose of determining the relative intensities of the various beta rays. Appropriate mass absorption coefficients measured by the authors [14]) were used for different beta rays.

In a very few cases, e.g. Sc^{45}, Ge^{76} and Sb^{123}, the resultant radioactive nucleus

was left not only in its ground state, but also in the excited isomeric state. For most of such cases the isomeric state decays by the isomeric transition to the ground state of the radioactive nucleus. The decay of the radioactive nucleus from the ground state, therefore, does not follow the simple decay law, but instead it follows the activity of the mixture due to both states. In such cases the disintegration rate of the ground state of the radioactive nucleus at any time t is given as

$$\lambda_b n_b = \frac{\sigma_a n_0 I \lambda_b}{(\lambda_b - \lambda_a)} (1 - e^{-\lambda_a t_e})(e^{-\lambda_a t} - e^{-\lambda_b t})$$

$$+ (\sigma_a + \sigma_b) n_0 I (1 - e^{-\lambda_b t_e}) e^{-\lambda_b t} + \frac{\sigma_a n_0 I \lambda_b}{\lambda_b - \lambda_a} (e^{-\lambda_b t_e} - e^{-\lambda_a t_e}) e^{-\lambda_b t},$$

where σ_a and σ_b are the cross-sections for the formation of the states a and b, λ_a and λ_b the decay constants of the states a and b (a is the isomeric state and b the ground state), n_0 is the number of nuclei in the target nucleus, I the incident flux, t_e the irradiation time and t the time which lapses between the stopping of irradiation and the start of the counting. For the cases when $\lambda_a \ll \lambda_b$ (e.g. Sc^{44m}) or when $\lambda_b \ll \lambda_a$ (e.g. Ge^{75m} and Sb^{122m}), the above equation may be reduced to simpler forms.

The half lives of various activities used in calculating the observed cross-sections were those actually observed by the authors. They have been found in general to tally with those quoted in the literature.

Errors in measurements of cross-sections may be introduced by five factors:

(1) *Error in the neutron flux measurement.* Since the cross-sections were not measured absolutely, but relatively, the accuracy in the measurement of $(nv)G$ (flux × geometrical efficiency) will depend upon the reproducibility of the geometry and the constancy of the flux. The error introduced in the cross-section measurements due to this factor was estimated to be about 2 % in most of the cases; but could go up to 5 % in some cases.

(2) *Error in the total number of nuclei* (n_0). An error in the estimation of n_0 may be introduced by the presence of impurities and by weighing. All the substances used were spectroscopically pure with purities generally more than 99.9 %. The overall error in n_0 was estimated to be about 0.5 %.

(3) *Error due to the presence of thermal neutrons and inelastically scattered neutrons.* The presence of thermal neutrons, even in small numbers, may cause serious errors, due to the high values of cross-sections at these energies. The source of these thermal neutrons would be the hydrogenous material around the target. Similarly the walls of the laboratory and other heavy masses around the target will give rise to inelastically scattered neutrons. In our case, the floor, which was nearer than the walls, was 150 cm from the target and no hydrogenous material was put near the target. The tritium target itself was mounted on a thin backing of brass of 0.001 cm thickness. The contribution

from these low energy neutrons is therefore expected to be quite small.

(4) *Statistical error.* The measurements in very many cases were repeated to keep the statistical error to about 3 %. In a few cases of high counting rate the statistical error was as small as 2 % and in cases of low counting rate it was as high as 8 %.

(5) *Miscellaneous errors.* Errors in these measurements may also be introduced by the uncertainties in the measurements of the times t_e and t. These times could be measured to better than 1 % accuracy. The detection of gamma rays by the beta counter is another source of error. The error due to these sources was estimated to be less than 3 %. Error due to the presence of D—D neutrons was negligible, since the threshold for (n, 2n) reactions is much above the energy of the D—D neutrons.

4. Results

In table 1 are given the cross-sections measured by the authors along with the values for some cases found in literature. The errors quoted by us include

TABLE 1

Observed values of (n, 2n) cross-sections at 14.8 MeV

Target nucleus	Observed half life	Observed cross-section (mb)	Paul and Clarke [1]	Forbes [2]
Cl^{35}	33.0 min	12±15%	3.47±45%	
K^{39}	7.7 min	6±15%	10.0 ±55%	
Sc^{45} [a]	4.01 h	148±15%		
Cr^{50}	41.6 min	27±25%		
Ga^{69}	67.5 min	1070±10%	552 ±30%	
Ga^{71}	21.0 min	2180±10%	700 ±15%	
Ge^{70}	41.0 h	1600±15%	666 ±35%	
Ge^{76}	78.0 min	1200±20%	1820 ±30%	
Sr^{84}	33.0 h	1770±10%		
Mo^{100}	67.0 h	1910±10%	3790 ±55%	
Ag^{107}	24.2 min	657±15%	519 ±50%	560±10%
Ag^{109}	2.3 min	710±15%	311 ±50%	1000±10%
Cd^{116}	43.5 d	490±15%		
	53.5 h	690±15%		
Sb^{121} [a]	16.5 min	1180±15%	750 ±25%	
Sb^{123} [a]	2.8 d	1950±10%	1245 ±25%	
Pr^{141}	3.5 min	1378±15%	2050 ±35%	
Eu^{151}	13.4 h	640±10%		
Eu^{153}	9.3 h	164±15%		
Gd^{160}	17.4 h	1725±10%	1470 ±55%	
Ho^{165}	38.5 min	2100±10%		
Yb^{176}	99.0 h	786±10%		
Re^{187}	89.0 h	1675±10%		
Ir^{191}	3.1 h	307±15%		

[a]) The values of these cross-sections are slightly different from these reported earlier [11]) because the adopted value for our standard reaction Fe^{56} (n, p) Mn^{56} is now 126 mb instead of 110 mb as used previously.

the statistical as well as the estimated errors arising from other sources as discussed earlier. As is evident from table 1 an appreciable difference exists in many cases between the values measured by the authors and those found in the literature.

5. Discussion

On the basis of the compound nucleus model, the (n, 2n) cross-section can be considered as the product of the cross-section $\sigma(n, n')$ for the first neutron being emitted (i.e. inelastically scattered) times the probability P_n that the second neutron will be emitted:

$$\sigma(n, 2n) = \sigma(n, n')P_n.$$

Assuming that the probability that the first emitted neutrons will have an energy between 0 and $\varepsilon_n - B_n$ is equivalent to the probability that the second neutron will be emitted, P_n has the value [15])

$$P_n = \frac{\int_0^{\varepsilon_n - B_n} \varepsilon_{n'} \sigma_c(\varepsilon_{n'}) W_y(\varepsilon_n - \varepsilon_{n'}) \, d\varepsilon_{n'}}{\int_0^{\varepsilon_n} \varepsilon_{n'} \sigma_c(\varepsilon_{n'}) W_y(\varepsilon_n - \varepsilon_{n'}) \, d\varepsilon_{n'}};$$

here ε_n is the incident neutron energy, B_n the binding energy of the neutron in the target nucleus, $\varepsilon_{n'}$ the energy of the first emitted neutron, $W_y(\varepsilon_n - \varepsilon_{n'})$ the level density of the target nucleus Y at the excitation energy $(\varepsilon_n - \varepsilon_{n'})$ and $\sigma_c(\varepsilon_{n'})$ the cross-section for the formation of the compound nucleus by the inelastically scattered neutron. The values of $\sigma_c(\varepsilon_n)$ were taken from the Fast Neutron Data Report [21]) and the binding energy of the neutron B_n for different targets from various sources [21–26]).

Since $\sigma(n, n') \approx \sigma_c(n)$ we have used the value of $\sigma_c(n)$ instead of $\sigma(n, n')$ for the actual computation of (n, 2n) cross-sections.

Treating the nucleus as a Fermi gas of nucleons, Lang and Le Couteur [9]) have shown that the spacing of nuclear levels of spin 0 in a nucleus of mass number A and excitation energy U is given by

$$\frac{1}{W_0} = D_0 = 0.11 A^{\frac{2}{3}}(U+t)^2 \exp - \left[2\left(\frac{AU}{11}\right)^{\frac{1}{2}} + \tfrac{3}{32}(11U)^{\frac{1}{3}}\right], \qquad (1)$$

where

$$t = (10.5U/A)^{\frac{1}{2}} - 7.9/A.$$

The following empirical relation between the level densities of odd, odd-mass and even nuclei, pointed out by Brown and Muirhead [16]), was used:

$$\tfrac{1}{12} W_0 \approx \tfrac{1}{5} W_{o.p.} \approx W_e.$$

We have also calculated the cross-sections for (n, 2n) reactions making use of the level density formula of Newton [10]), which takes into account the shell effects. For the spacing between the energy levels having total angular mo-

mentum 0, at an excitation energy U, of a Fermi gas model of a nucleus containing Z protons and N neutrons, Newton obtains

$$D_0 = A^{\frac{1}{3}}(2\tilde{j}_N+1)^{\frac{1}{2}}(2\tilde{j}_Z+1)^{\frac{1}{2}}(2U+3t)^2 \times \exp\{8.75-0.4982(\tilde{j}_N+\tilde{j}_Z+1)^{\frac{1}{2}}A^{\frac{1}{3}}U^{\frac{1}{2}}\} \quad (2)$$

where

$$t = (6\pi^{-2}G^{-1}U)^{\frac{1}{2}}, \qquad G = G_1+G_2 = 2\alpha(\tilde{j}_N+\tilde{j}_Z+1)A^{\frac{1}{3}},$$
$$G_1 = \alpha(2\tilde{j}_N+1)A^{\frac{1}{3}}, \qquad G_2 = \alpha(2\tilde{j}_Z+1)A^{\frac{1}{3}}.$$

TABLE 2

Observed and calculated values of (n, 2n) cross-sections

Target nucleus	Observed cross-section σ_{obs} (mb)	σ_{cal}(mb) (eq.(1))	σ_{cal}(mb) (eq.(2))	B_n	$\frac{\sigma_{obs}}{\sigma_{cal}}$ (eq.(1))	$\frac{\sigma_{obs}}{\sigma_{cal}}$ (eq.(2))
Cl^{35}	12 ±15%	60	52	13.2	0.2	0.23
K^{39}	6 ±15%	29	14	13.2	0.21	0.43
Sc^{45}	148 ±15%	410	897	11.05	0.36	0.165
Ti^{46}	50.4±16% [19])	58	71	13.3	0.867	0.71
Cr^{50}	27 ±25%	34	49	13.4	0.795	0.55
Co^{59}	149 ± 5% [18])	1170		10.25	0.128	
Ni^{58}	40.6±30% [18])	625	360	11.7	0.06	0.11
Cu^{63}	510 ± 7% [2])	1000	986	10.65	0.51	0.516
Cu^{65}	970 ± 8% [2])	1050		9.97	0.92	
Zn^{64}	224 ±20% [1])	675		11.72	0.33	
Ga^{69}	1070 ±10%	1255	970	10.1	0.85	1.1
Ga^{71}	1950 ±10%	1430	1010	9.05	1.36	1.93
Ge^{70}	1600 ±15%	900	840	11.6	1.78	1.9
Ge^{76}	1200 ±20%	1450	1500		0.825	0.8
As^{75}	545 ±29% [1])	1350		10.2	0.42	
Se^{82}	1500 ±23% [1])	1700		9.8	0.88	
Br^{79}	1141 ±25% [1])	1320		10.65	0.86	
Zr^{90}	80 ±50% [1])	940	680	12.3	0.085	0.12
Mo^{100}	1910 ±10%	1690	1650	8.9	1.13	1.15
Ru^{96}	478 ±19% [1])	1800	1800	8.6	0.265	0.265
Ag^{107}	657 ±15%	1750	1880	10.3	0.375	0.35
Ag^{109}	710 ±15%	1850		9.07	0.38	
Cd^{116}	1180 ±15%	2030	2020	8.5	0.58	0.58
Sb^{121}	1180 ±15%	1950		9.3	0.61	
Sb^{123}	1950 ±10%	2000	2120	9.3	0.975	0.92
Te^{130}	594 ±20% [1])	2000	2180	8.25	0.3	0.27
I^{127}	1120 ±35% [1])	2100		9.28	0.53	
Pr^{141}	1378 ±15%	2000	2170	9.44	0.65	0.60
Sm^{144}	1200 ±25% [3])	2300			0.52	
Sm^{154}	1500 ±20% [3])	2400		8.00	0.63	
Eu^{151}	640 ±10%	2320	2400	8.15	0.275	0.266
Eu^{153}	164 ±15%	2330		7.14	0.069	
Gd^{160}	1725 ±10%	2400	2410	7.2	0.718	0.72
Ho^{165}	2100 ±10%	2500	2520	8.1	0.84	0.83
Yb^{176}	786 ±10%	2650	2650	6.85	0.296	0.296
Re^{187}	1675 ±10%	2680	2690	7.3	0.62	0.62
Au^{197}	1722 ±27% [1])	2700		8.05	0.64	

Here, $\bar{j}_Z$ and $\bar{j}_N$ are effective values of j for the protons and neutrons in the ground state, defined by Newton [10]) and G_1 and G_2 the individual neutron and proton densities. As the effective values of j for the various nuclei at the actual excitation energies were not known, W_0 was used instead of W_J in the calculation of (n, 2n) cross-sections, using the level densities (1) and (2). The values so calculated, along with the observed values, are shown in table 2.

The measured values of (n, 2n) cross-sections are plotted in fig. 1 against the neutron number. Also included are the values of (n, 2n) cross-sections from the literature for those cases in which we have not repeated the measurements.

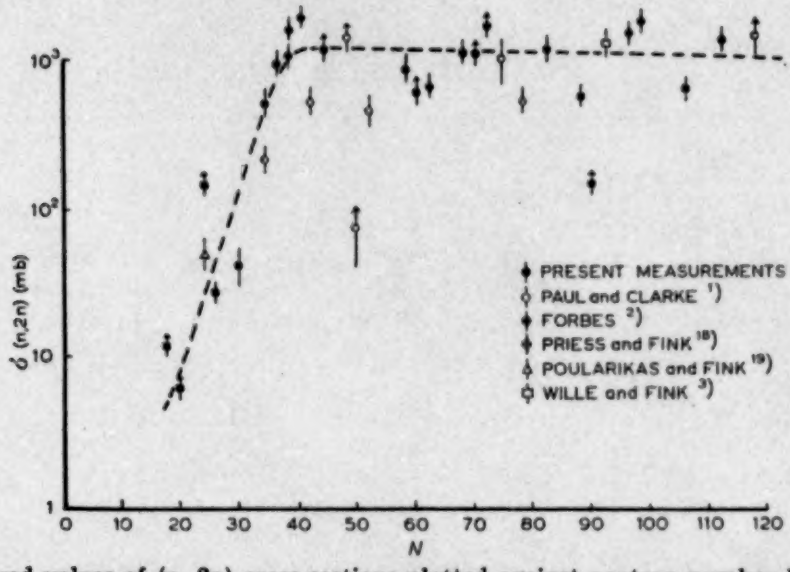

Fig. 1. Measured values of (n, 2n) cross-sections plotted against neutron number N. The dotted line is an arbitrary line, drawn to show the trend.

The measured values at first increase steeply with the neutron number from values ≈ 6 mb at $N = 20$ to about 1200 mb at $N = 40$. From $N = 40$ to $N = 130$, the measured values fluctuate around a mean value of 1200 mb. This trend is shown by an arbitrary dotted line. A few points, at $N = 19$, 50, 60, 70, 90 and 114, lie quite below the general trend. They correspond to the reactions $K^{39}(n, 2n)K^{38}$, $Zr^{90}(n, 2n)Zr^{89}$, $Ag^{107}(n, 2n)Ag^{106}$, $Sb^{121}(n, 2n)Sb^{120}$, $Eu^{153}(n, 2n)Eu^{152}$ and $Ir^{191}(n, 2n)Ir^{190}$, respectively. Each of the resulting radioactive nuclei K^{38}, Zr^{89}, Ag^{106}, Sb^{120}, Eu^{152} and Ir^{190} is reported in the literature [17]) to have two independent half lives. Since the cross-section for only one of the activities in each case was measured, the measured values for these isotopes are lower than the actual values and accordingly lie below the general trend.

In figs. 2(a) and 2(b) are plotted the ratios of the observed (n, 2n) cross-sections to the calculated ones against the neutron number, using the level density formulas of Lang and Le Couteur [9]) and Newton [10]). The dotted lines

are arbitrary lines drawn to show the general behaviour of the ratios. Two general results arise from the study of these curves.

Almost all points except a few lie below unity. For some of the points, the

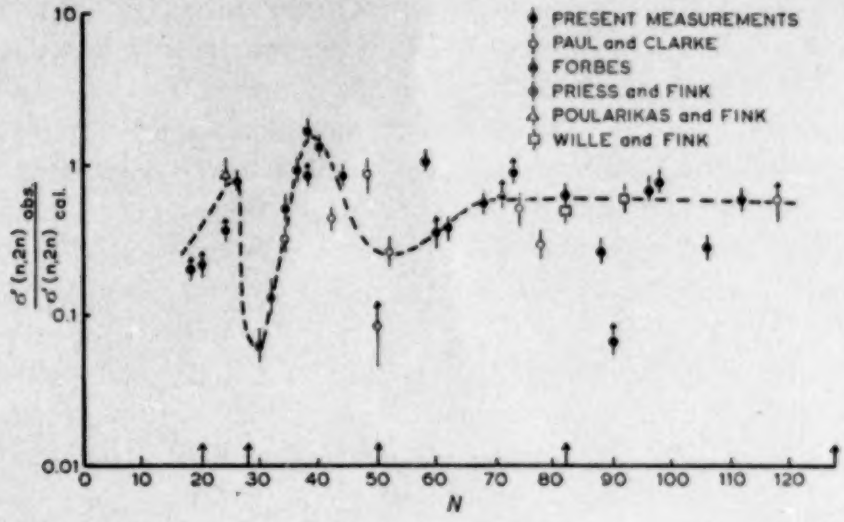

Fig. 2(a). The ratio of the observed and calculated values of (n, 2n) cross-sections is plotted against neutron number N. The points with arrows are the lower limits for observed values and should actually be raised since only one of the isomers was observed. The calculated values are based on the level density formula of Lang and Le Couteur [9]). The dotted line is an arbitrary line to show the trend.

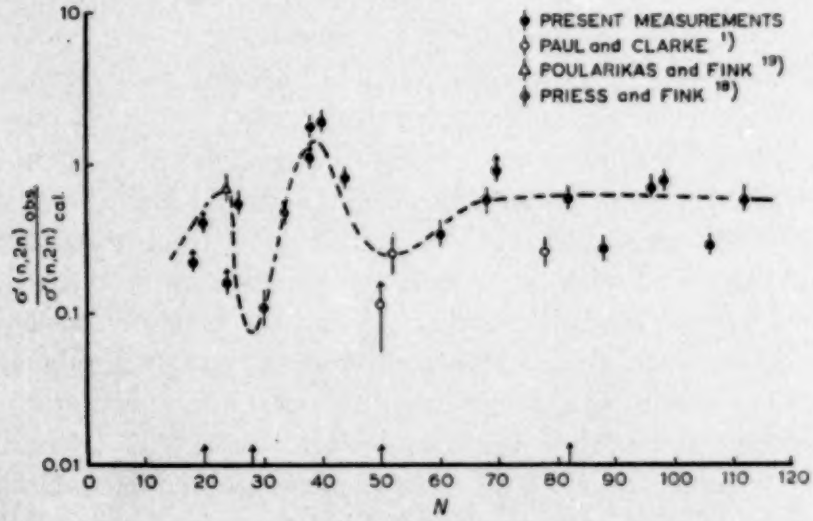

Fig. 2(b) The same as fig. 2(a), but the calculated values are based on the level density formula of Newton [10]).

ratios are expected to be low due to the fact that the cross-section for only one of the two possible half lives was measured as explained earlier. For the rest of the cases, however, there seems to be a real discrepancy between the theoretical and the observed values. This may be due to various causes:

(1) The cross-section for the formation of the compound nucleus $\sigma_c(n)$ was used instead of $\sigma(n, n')$ in the calculation of $\sigma(n, 2n)$ which may give higher calculated values of $\sigma(n, 2n)$. At this energy it is safe to assume that $\sigma_c(n) = \sigma(n, n')+\sigma(n, p)+\sigma(n, \alpha)+$some other cross-sections which may be neglected. In the light of the known values of $\sigma(n, p)$, $\sigma(n, \alpha)$ and $\sigma(n, n')$ the discrepancy between $\sigma(n, n')$ and $\sigma_c(n)$ is expected to be less than 10 % for light elements and less than 1 % for medium and heavy elements.

(2) The level density formulae used in the calculations were for the levels with $J = 0$. In actual practice, however, the levels at the various excitation energies may have some other values of J; the dependence of the average J on the excitation energy is not known. The level density at $J = 0$ is related to the level density at $J \neq 0$ by the approximate formulae $W(U, J) = W_0(U)(2J+1)/2\sigma^2(U)$, where $\sigma^2(U)$ is in principle a function of the excitation energy U (ref. [20])). In this way, $W(U, J)$ may have a different energy dependence than that assumed in the calculation of $\sigma(n, 2n)$. But as this calculation involves the ratio of two quantities, each involving $W(U, J)$, the discrepancy due to this cause is expected to be only a few percent.

(3) As suggested by Weisskopf [6]), the most probable cause of such a discrepancy may be that the compound nucleus may sometimes decay before the thermal equilibrium between the various nucleons in the nucleus has been achieved. The neutron emitted under these conditions will generally have higher energy than those emitted after the formation of compound nucleus, leaving behind a smaller energy for the excitation of the residual nucleus, which may therefore not be able to emit a second neutron. The calculated values based on the assumption of the decay of the compound nucleus will, therefore, tend to be higher than the measured values.

There is evidence of minima in the ratio of $\sigma_{obs}/\sigma_{cal}$ at $N = 28$ and 50. These minimas seem to be associated with the shell effects. Near or at magic numbers, the level density is expected to be low. With this in mind, the observed cross-sections were compared with the calculated values using the level density formula of Newton [10]), as described earlier, so that the shell effects are taken into account. Fig. 2(b) gives the ratio of $\sigma_{obs}/\sigma_{cal}$ against neutron number in this case. We still have the minima at roughly the same value of neutron number, though the values of the ratios at the minima are raised a little. This effect may also be explained by assuming that the decay of the compound nucleus before thermal equilibrium is reached may be more probable for magic or near magic number cases, since the unoccupied higher levels may then be so far apart that the energy of the incoming neutron may be shared only by a few nucleons: the emission of one of these comparatively high energy nucleons may leave the nucleus less excited than required for the emission of the second neutron.

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"SELF-SCINTILLATION" STUDY OF THE BETA DECAY OF Rb^{87}

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Abstract: The beta decay of Rb^{87} has been studied using the self-scintillations produced in three NaI(Tl) crystals doped with rubidium. The beta spectrum was measured down to 5—8 keV. An end point energy of 274 ± 3 keV and a specific activity equal to 780 ± 14 s^{-1} per gram of natural rubidium were found. Assuming an isotopic abundance of 27.85% for Rb^{87}, a half-life of $(5.53 \pm 0.10) \times 10^{10}$ y is obtained.

1. Introduction

The long-lived beta activity of Rb^{87} has been of interest for some time [1–16]). Since the transition is from the $\frac{3}{2}-$ state of Rb^{87} to the $\frac{9}{2}+$ ground state of Sr^{87}, it is classified as non-unique third-forbidden. The half-life is of particular interest because of its potential usefulness in determining geological ages from the ratio of radiogenic Sr^{87} to Rb^{87}. However, owing to the low specific activity and low energy of the beta particles, it has been difficult to make precise measurements of this half-life and there has been considerable disagreement in the values quoted, ranging from about 4×10^{10} to 6×10^{10} y. Table 1 contains a summary of the published results of half-life determinations. Also listed are the authors and the methods they used. It is of interest to note that these half-life values fall into two groups, one in the range from 4.3×10^{10} to 5.1×10^{10} y and the other in range from 5.8×10^{10} to 6.5×10^{10} y. Apparently there is no correlation among the results of any particular method, with the possible exception that the two half-life determinations made using self-scintillations produced in RbI(Tl) lie within the upper range. However, one should also note that the value obtained using the self-scintillations in a Rb loaded liquid scintillator lies in the lower range.

Preston *et al.* [17]) have made an analysis of the general beta spectrum shape using the results of Goodman [18]). While there is not complete agreement in the experimental shapes obtained in previous investigations the F-K plots lie close enough together to give agreement with the analysis by Preston *et al.* above an energy of 40 keV.

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TABLE 1

Half-life of Rb^{87} beta decay

Method	Rb^{87} half-life ($\times 10^{10}$ y)	Author (year)
Cloud chamber	4.5	G. Orban [1] (1931)
Sr^{87}/Rb^{87} in minerals of ages determined from U/Pb	6.3	F. Strassman and E. Walling [2]) (1938)
Gas counter	5.8 ±1.0	S. Eklund [3]) (1946)
Gas counter	6.5 ±0.6	O. Haxel *et al.*[4]) (1948)
Gas counter	6.0 ±0.6	M. Kemmerich [5]) (1949)
High pressure gas counter	6.2 ±0.3	S. C. Curran *et al.*[6]) (1951)
RbI(Tl) scint. crystal	5.9 ±0.3	G. M. Lewis [7]) (1952)
Gas counter	6.2 ±0.2	J. Flinta and S. Eklund [8]) (1954)
Gas counter	4.3 ±0.4	I. Geese-Bahnisch and E. Huster [9]) (1954)
Gas counter (enriched Rb^{87})	6.2 ±0.3	M. H. MacGregor and M. L. Wiedenbeck [10]) (1954)
Sr^{87}/Rb^{87} in minerals of ages determined from U/Pb	5.0 ±0.2	L. T. Aldrich *et al.*[11]) (1956)
Sr^{87}/Rb^{87} in minerals of ages determined from A^{40}/K^{40}	4.6 ±0.5	K. Fritze and F. Strassman [12]) (1956)
Gas counter	5.07±0.20	W. F. Libby [13]) (1957)
Rb doped liquid scint.	4.70±0.05	K. F. Flynn and L. E. Glendenin [14]) (1959)
Gas counter	4.72±0.08	W. Rausch and W. Schmidt [15]) (1960)
RbI(Tl) scint. crystal	5.82±0.1	K. Egelkraut and H. Leutz [16]) (1961)
NaI(Tl, Rb) scint. crystal	5.53±0.10	Present work (1961)

The investigation described here was initially undertaken in an effort to obtain a beta spectrum reliable to lower energies than had hitherto been possible and to make a new measurement of the specific activity †.

2. The Self-Scintillating Crystals

NaI(Tl, Rb) crystals (i.e., rubidium doped NaI(Tl) crystals) were grown with 1% by weight of RbI added to the sodium iodide melt ††. Since the end product was somewhat uncertain as to the rubidium content, sections of the ingot were saved from both immediately above (a 1.9 cm diam. by 0.63 cm high disk) and below (1.9 cm diam. by 0.63 cm high cone) the centre section. This centre section consisted of a 2.5 cm high by 1.9 cm diam. cylinder. The top and bottom sections were studied unmounted, whereas the central section was mounted in a thin aluminium can in the usual manner. An undoped NaI(Tl) crystal, 2.5 cm high and 1.9 cm diam., was grown under the same conditions and mounted in the same way. This undoped crystal was used for comparison measurements and for all background determinations.

† A preliminary account of this work has been given earlier [25]).

†† These crystals were grown by the Levinthal Electronic Products, Inc. Palo Alto, California.

The Rb^{87} counting rate per gram of crystal was determined for each of the three sections. The resulting values differed considerably and showed that the rubidium did not distribute uniformly throughout the ingot. The conical tip where the ingot started contained about 0.1 % RbI while the mounted crystal had 0.2 % and the disk contained 0.4 %. Considering that Rb^+ has an ionic radius of 1.49 Å while that of Na^+ is only 0.98 Å, it is not surprising that the rubidium segregated out in large measure. Recent work on the distribution of Tl in NaI(Tl) crystals [19]) shows that it is also not distributed uniformly.

To see whether the non-uniform distribution of the Rb had any undesirable effects on the scintillation properties of the doped crystals, their scintillation properties were compared in some detail with those of the comparison crystal. The scintillation efficiencies of the three Rb doped crystals were found to be about 80 % of the result obtained using the undoped crystal. The values for the cone and disk were slightly higher and lower, respectively, than that of the mounted section, indicating that the scintillation quenching is dependent on the relative rubidium content. For the 662 keV Cs^{137} gamma rays, the resolution of the mounted doped crystal was 13.6 % as compared to 12.4 % for the undoped comparison crystal. Measurements of the pulse-height spectra of collimated Cs^{137} gamma rays incident on the side of the mounted crystals, normal to the cylinder axis, were made as a function of position above the photomultiplier. A slight effect on the position of the photopeak was noted which was, however, the same for both the doped and undoped crystals and hence was ascribed to geometry effects rather than crystal composition.

The pulse-height, as a function of gamma ray energy, was measured for energies in the range from 0.030 to 1.33 MeV with the mounted crystal. The gamma and X-ray sources were placed approximately 10 cm vertically above the crystal in each case. Similar tests with the two unmounted sections were made in the range from 0.008 to 0.662 MeV. In all cases the linearity was found to be better than 2 % for energies above 80 keV. Below this energy X-rays were used and linearity was obtained to the accuracy that the average X-ray energies were known (4—5 %). Since the mean free paths of these different gammas in sodium iodide vary from 10^{-3} mm to 5 cm, this also implies that the non-uniformity of Rb concentration along the length of the crystals does not adversely affect the scintillation response to a significant degree. We therefore concluded that the position in which a beta particle of given energy is emitted by a Rb^{87} nucleus would have relatively little effect on the amplitude of the resultant photomultiplier pulse and, consequently, little effect in distorting the beta spectrum.

Optical transmission measurements were made on the doped pieces and compared with those made on undoped NaI(Tl) crystals. No differences were found in the wavelength region in which the photomultipliers are sensitive [20]).

3. Results

The beta counting was performed in a shielding arrangement that has been previously described [21]). No anticoincidence arrangement was used. The integral counting was performed with as many as four completely different counting systems: two single-channel analyzers and two multi-channel analyzer systems. The counting rates obtained with the different systems agreed to within

TABLE 2

Results of the chemical analyses and integral counting on the NaI(Tl, Rb) crystals

Crystal	Mass (g)	Total Rb content (mg)	Average Rb^{87} integral counting rate (counts/min)	Rb^{87} specific activity (counts/min per mg Rb)	Rb^{87} half-life ($\times 10^{10}$ y)
Disk (1.9 cm diam. $\times$0.63 cm high)	6.763	10.03$\pm$0.10	472$\pm$9[a])	46.8$\pm$1.0	5.53$\pm$0.12
Cone (1.9 cm diam. $\times$0.63 cm high)	2.874	0.944$\pm$0.02	438$\pm$18[a])	46.4$\pm$1.9	5.58$\pm$0.23
Mounted section (1.9 cm diam. $\times$2.54 cm high)	26.76	14.29$\pm$0.14	671$\pm$10[b])	47.0$\pm$0.8	5.51$\pm$0.10
			Weighted average		5.53$\pm$0.10

[a]) Data taken with one single-channel system and one multi-channel system.
[b]) Data taken with two separate single-channel systems and two separate multi-channel systems.

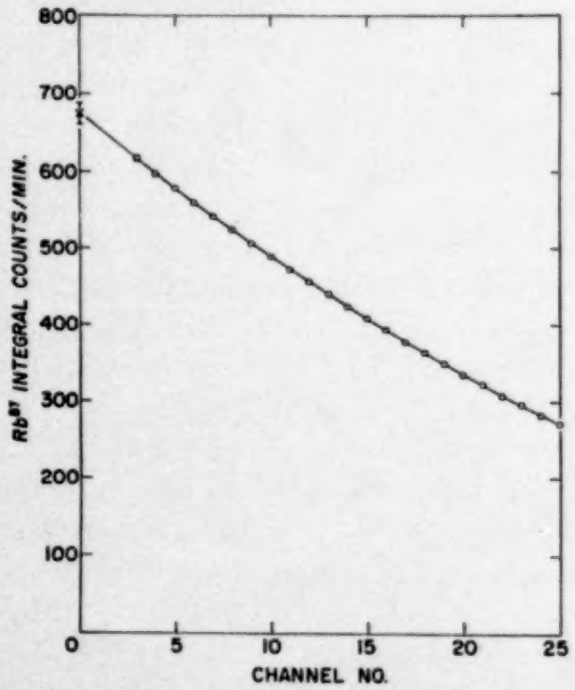

Fig. 1. Rb^{87} integral spectrum. These data were taken with the mounted NaI(Tl, Rb) crystal and one of the multi-channel pulse-height analyzer systems. The extrapolation to zero energy is linear from approximately 5 to 6 keV.

$\pm$ 2 %. Table 2 contains the average values of the integral counting rates obtained for each crystal. The background counting rate was determined using the mounted undoped comparison NaI(Tl) crystal. A factor to normalize the background measured with this crystal to that detected by the disk and cone was found by comparing the counting rates above the beta end point. No such factor was needed for the mounted crystal. The data presented in table 2 have been corrected for background. Fig. 1 shows a typical integral spectrum. This particular spectrum was taken with an EMI 9578 S photomultiplier, the mounted crystal and a Nuclear Data model 102 transistorized 256-channel pulse-height analyzer.

Quantitative chemical analyses were performed on each of the three doped crystals at the Argonne National Laboratory after the counting was completed. A detailed description of these analyses will be published elsewhere [22]).

The rubidium content was determined by precipitating rubidium as the $Rb\Phi_4B$ compound and weighing as such. The crystal was first dissolved in H_2O, treated with HNO_3 to remove I_2 and then converted to NaCl by evaporation to dryness with HCl. Thallium which is also precipitated by the precipitant, sodium tetraphenylboron, was removed using anion exchange prior to the precipitation of the rubidium. In the analyses determinations were made on duplicate aliquots of the unknown solution as well as on standard solutions.

It was necessary to correct the rubidium value first obtained for small amounts of potassium and cesium which are also precipitated in the method of analysis used. The amount of potassium contamination was determined using a flame photometer and the potassium content was found to be not more than a few per cent by weight relative to the Rb content in each case. It is most likely that this potassium was introduced as an impurity in both the NaI and RbI. The contribution to the measured beta activity by the K^{40} betas from this contamination was completely negligible. An estimate of the cesium contamination present was made spectrographically and found to be such as to contribute an error of only about 0.25 % in the Rb determination.

Considering the various corrections that had to be made, it is felt that the Rb analyses are accurate to within 1 %. The results of the analyses are given in table 2.

The Rb^{87} weighted average half-life obtained with the three crystal sections is $5.53 \pm 0.10 \times 10^{10}$ y. This assumes an isotopic abundance of 27.85 % for Rb^{87}. The error includes the uncertainty in the Rb analysis as well as that of the counting rate determination.

4. Spectrum Shape

Fig. 2 shows an uncorrected beta spectrum obtained using the 2.5cm $\times$ 1.9 cm diam NaI(Tl, Rb) crystal. Also shown in fig. 2 is the background obtained

with the comparison crystal. Data were taken at normal amplifier gain and also at twice normal gain in order to observe the lower energy region in more detail. Fig. 3 is the Fermi-Kurie plot obtained. Corrections for finite energy resolution [23]) have been made. These corrections were minor except near the end point. The estimated correction for the escape of Rb^{87} betas [24]) through the surface before losing all their energy was smaller than the statistical error over most of the spectrum.

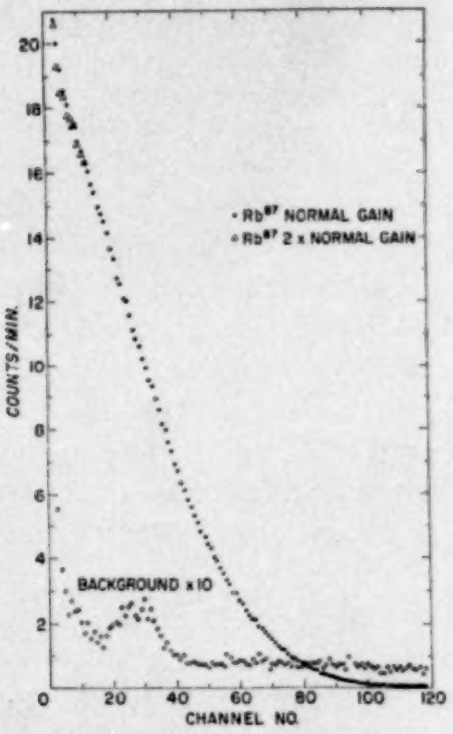

Fig. 2. Beta spectrum of Rb^{87} and background. These data are uncorrected. The solid circles show the data taken at normal amplifier gain and the triangles show those taken at 2 times normal gain.

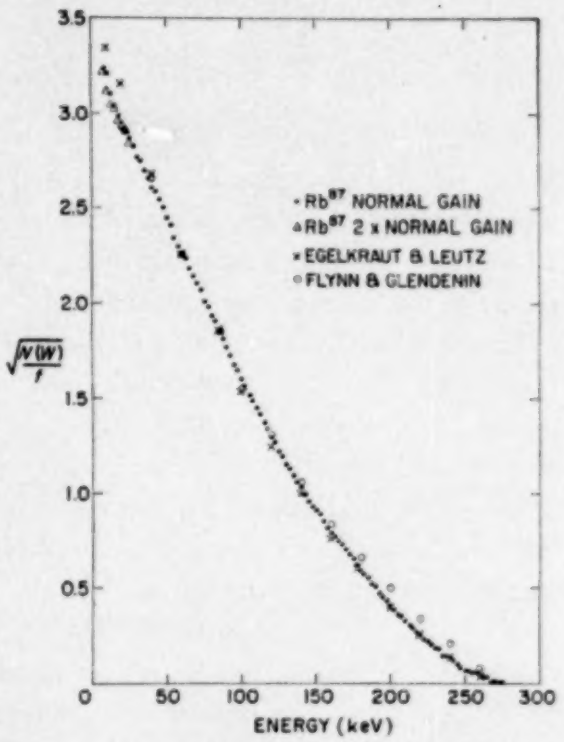

Fig. 3. Rb^{87} Fermi-Kurie Plot. The triangles are the points taken with 2 times normal amplifier gain. Shown for comparison are representative points from the F-K plots of Egelkraut and Leutz (×) and of Flynn and Glendenin (o) normalized at 80 keV.

Fig. 3 also shows representative points from the F-K plots published by Flynn and Glendenin [14]) obtained using a liquid scintillator loaded with rubidium and by Egelkraut and Leutz [16]) determined with a NaI(Tl, Rb) crystal. The plots are all normalized at the 80 keV point. It can be seen that the F-K plot obtained in this work is in good agreement with that reported by Egelkraut and Leutz down to around 40 keV. It is somewhat surprising that the deviation at the lower energies is so great as presumably neither electronic noise nor background should be important for energies above 5 to 10 keV. Egelkraut and Leutz present a figure containing F-K plots using RbI(Tl), CsI(Tl, Rb) and NaI(Tl, Rb) crystals together with the results of other

authors for comparison. Most of the published F-K plots, as well as the unpublished results of Goodman [18]), agree reasonably well at energies above 40 keV.

We estimate that our F-K plot is valid down to about 5-8 keV where electronic noise becomes a significant factor. The actual point where the noise becomes significant is uncertain because of the different scintillation efficiencies of the doped crystal and the comparison one, i.e., the signal-to-noise ratios of the two crystals are different.

5. Conclusions

Table 1 also contains the result of the half-life measurement described above. It can be seen that the value obtained in this investigation lies in between the two groupings of previously published results, although it favours the higher set. In general, we are unable to offer any suggestions to explain the differences between the various results. However, a possible explanation of the difference between our half-life value and that of Egelkraut and Leutz could be as follows: Highly purified RbI is hard to prepare and a total contamination of other alkali metals, Na, K and Cs, of as little as $1\frac{1}{2}$ % in the RbI used by them could account for the difference in the half-life values to within the limits of error.

The general shape of the beta spectrum agrees with the shapes obtained in previous investigations and so should also be in agreement with the analysis carried out by Preston *et al.* [17]).

We wish to thank Mr. K. J. Jensen who performed the analyses for the rubidium content and to express our appreciation to the Argonne National Laboratory for making it possible for Mr. Jensen to do this work for us. The assistance of Mr. W. N. Schreiner and Mr, D. A. Gollnick in the recording of the data and the computations is acknowledged.

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THE s-WAVE AND p-WAVE STRENGTH FUNCTIONS

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Abstract: The s-wave and p-wave neutron strength functions, calculated with an optical model, are compared with experiment for spherical nuclei. The real potential is diffuse and has a spin-orbit term. The imaginary potential is surface-peaked. This surface-peaking produces experimentally required low strength functions between resonances. A detailed fit to the p-wave strength functions required parameters considerably different from those required to explain other data.

1. Introduction

In the last few years a large number of experiments have been performed which make possible the determination of s-wave neutron strength functions (average level widths divided by average level spacing) over an extensive region of the periodic table [1]). Recently there have also been determinations of the p-wave strength function, mainly in the region of mass number 80 to 120. Most of these p-wave strength functions are determined indirectly from the measurement of neutron capture cross sections [2] ††). There are also some direct measurements using discrete p-wave scattering resonances (Nb, Ag, Pb) [3]).

2. Discussion

It has been suggested [2]) that the measured p-wave strength function near $A = 100$ indicates a splitting of the p-wave resonance (which the optical model predicts in this region of A) into a $p_{\frac{3}{2}}$, $p_{\frac{1}{2}}$ doublet. We have obtained a fit to the p-wave strength function data (fig. 1) and then calculated the s-wave strength function by using the same parameters (fig. 2).†††

The following potential was used:

$$-V(r) = V_0 p(r) + iV_{01} q(r) + \frac{\alpha}{2r}\frac{\mathrm{d}p}{\mathrm{d}r} V_0 \begin{Bmatrix} l \\ -(l+1) \end{Bmatrix},$$

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†† There are also some preliminary data from the Oak Ridge group [7]).

††† Data in regions of A where nuclei are highly deformed have not been plotted. These regions require special consideration.

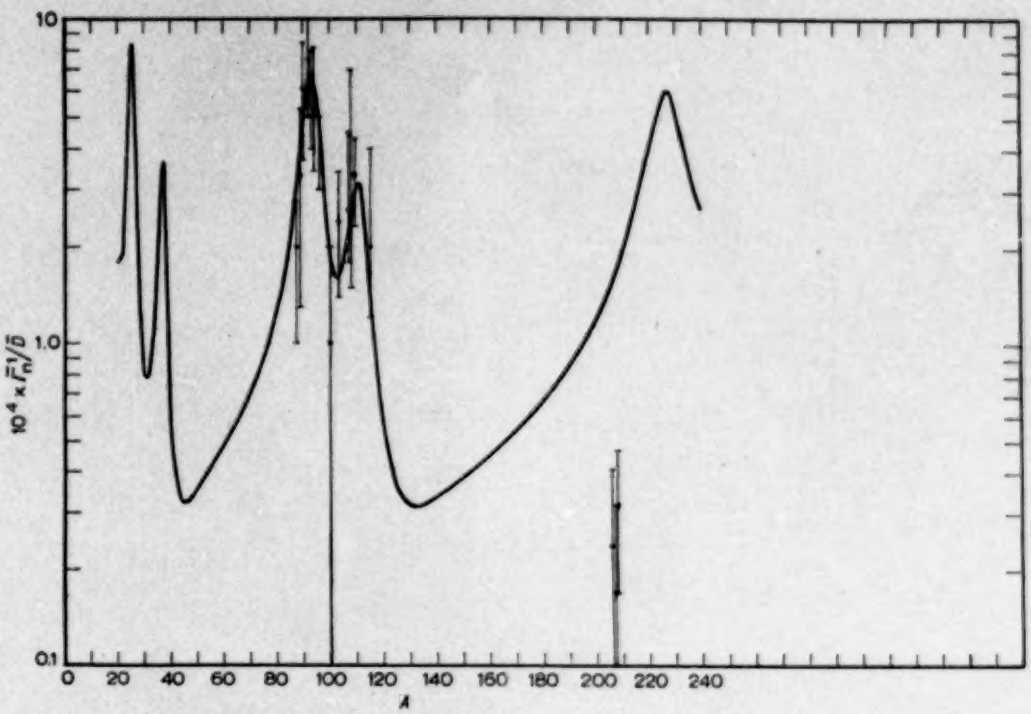

Fig. 1. Theoretical and experimental values of the p-wave strength function. The experimental data are from refs. [2, 3]).

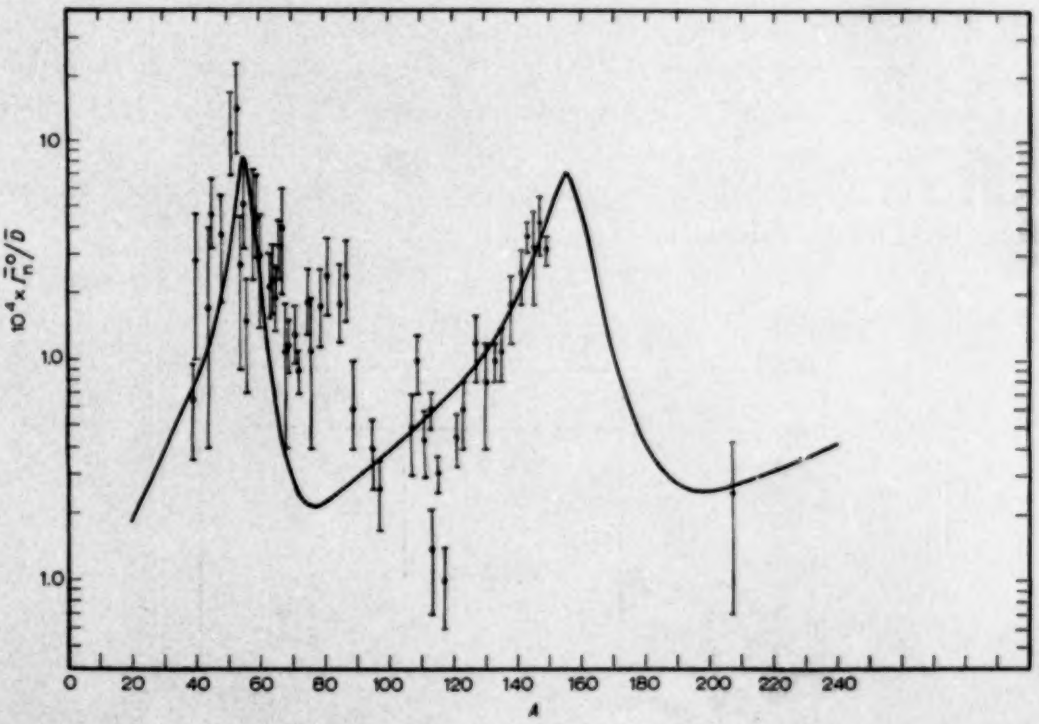

Fig. 2. Theoretical and experimental values of the s-wave strength function. There are measurements by Hughes *et al.*[1]) and by Coté *et al.*[1]). There are also data from ref. [3]) and from a compilation of s-wave strength functions communicated by J. H. Gibbons.

where

$$p(r) = \begin{cases} 1 & \text{for} \quad 0 \leqq r \leqq f, \\ \exp[-(r-f)^2/a^2] & \text{for} \quad r > f, \end{cases}$$

$$f = r_0 A^{\frac{1}{3}} - a(\ln 2)^{\frac{1}{2}},$$

and

$$q(r) = \begin{cases} 1 & \text{for} \quad 0 \leqq r \leqq c, \\ 1 + \dfrac{V_{02}}{V_{01}} \exp[-(r-R_1)^2/d^2] & \text{for} \quad c < r \leqq R_1, \\ \left(1 + \dfrac{V_{02}}{V_{01}}\right) \exp[-(r-R_1)^2/b^2] & \text{for} \quad r \geqq R_1, \end{cases}$$

with

$$c = r_0 A^{\frac{1}{3}} - 0.8 \text{ fm}.$$

The constant d is chosen so that

$$q(c) = 1 + 10^{-4} \frac{V_{02}}{V_{01}}.$$

The imaginary potential is peaked at R_1, given by

$$R_1 = r_0 A^{\frac{1}{3}} + 1.25 \text{ fm}.$$

Similar forms of imaginary potential, surface peaked in character, have been suggested by several authors [4]). The real potential was given the form above because it saves computer time compared to the Fermi distribution which is more commonly used. The results computed from the different shapes are not expected to vary strongly if they have the same surface thickness. The parameters used in the calculations are given in table 1.

TABLE 1

Values of parameters for the figures

	Figs. 1, 2	Figs. 3, 4
V_0 (MeV)	50.2	47
V_{0_1} (MeV)	0.6	0.6
V_{0_2} (MeV)	0.8	0.8
r_0 (fm)	1.3	1.25
a (fm)	0.833893	2.0
b (fm)	0.98	0.98
α (fm^2)	−1.45	−0.63

In order to split the p-wave resonance near $A = 100$ it has been found necessary to make the spin-orbit force considerably larger than is needed for the shell model and for higher energy scattering [5]) (about twice as large).

Moreover, to get a good fit to the data, it has been found necessary to have a surface term for the real part of the potential having a width about half the normally accepted value [5]). It is to be noted that resonance measurements of the p-wave strength function in the region of Pb do not fit the calculated curve.

From the paragraph above one sees that if one asks for a detailed fit to the p-wave strength function, parameters are required which are quite different from those which explain other data. Looking at the s-wave data, one sees that there is considerable deviation of theory from experiment in the region

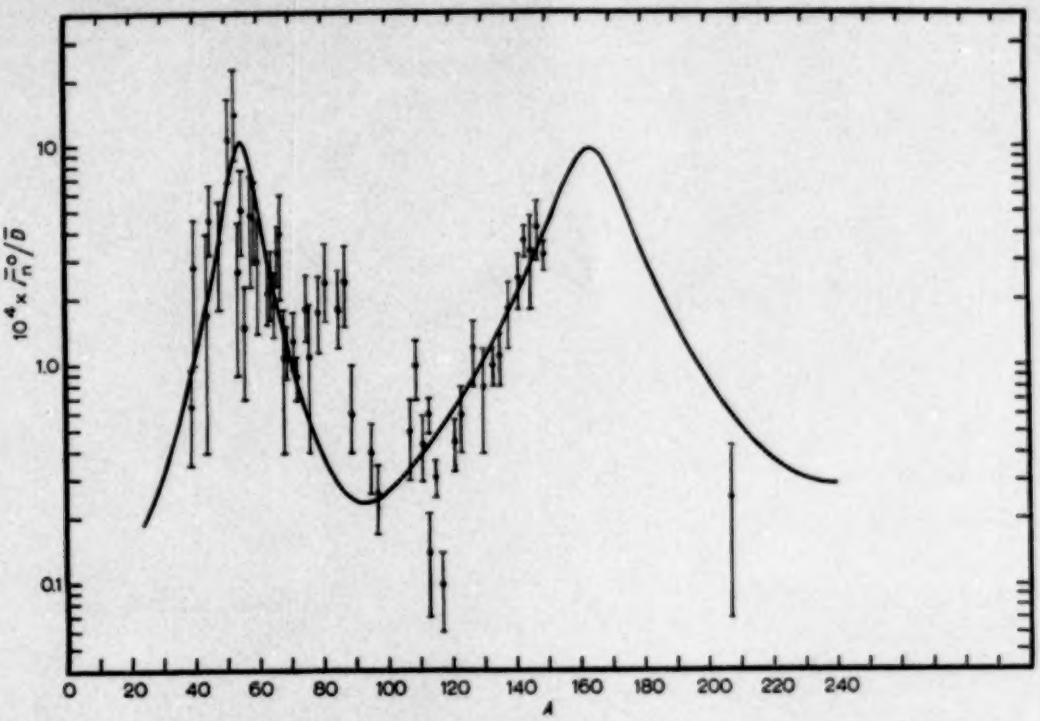

Fig. 3. Theoretical and experimental values of the s-wave strength function. The real potential has been changed from fig. 2 to improve the fit.

near $A = 80$. This also raises the question whether detailed fits to data as they exist now, using an optical model of the type described above, are significant over a small region of A.

In the past it has been found difficult to make optical model calculations giving a low enough s-wave strength function between resonances. The calculated curves in fig. 2 and fig. 3 show strength functions in these regions smaller than those of earlier calculations. The curve shown in fig. 3 was calculated by using a potential with normal surface thickness for the real potential. It is believed that the low strength function obtained in the region between resonances is a result of the form of the absorptive part of the potential used here which is surface peaked. It is to be noted that [6])

$$\frac{\overline{\Gamma_n^0}}{D} \propto \int_0^\infty |u|^2 W \mathrm{d}r,$$

where u/r is the radial wave function and W the imaginary part of the potential. This surface peaking of W is a strong indication of the effectiveness of the Pauli exclusion principle in low energy neutron scattering.

In fig. 4 we plot calculated p-wave strength functions using the normal spin-orbit force (see table 1) with the other parameters the same as those used in fig. 3.

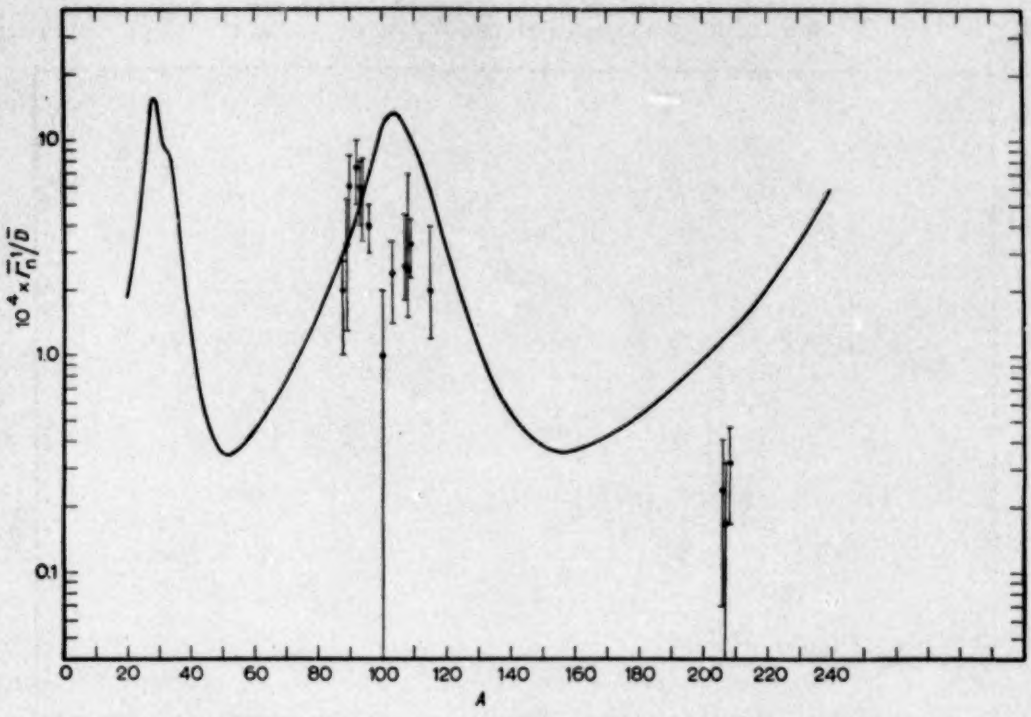

Fig. 4. Theoretical and experimental values of the p-wave strength function for normal spin-orbit coupling.

The authors wish to thank L. A. Bowman of the Aeronautical Research Laboratory for handling the computations and drawing the figures.

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THE SECOND-ORDER OPTICAL POTENTIAL FOR PION-NUCLEUS SCATTERING

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Abstract: The optical model for high energy elastic scattering of pions by nuclei may be explicitly evaluated in terms of two-nucleon correlations in nuclei (neglecting tertiary and higher order correlations), using dispersion relations to give the relevant scattering amplitudes. Previous treatments of the effects of pair correlations have assumed specific nuclear models. In the present discussion it is shown how this may be done avoiding model-dependent assumptions.

1. Introduction

Expressions for the optical model potential for pion-nucleus scattering in terms of pion-nucleon scattering amplitudes have been given by several authors [1, 2]). These have been of particular interest, since the pion-nucleon scattering amplitudes may be obtained directly from measured pion-nucleon scattering cross sections with the use of dispersion relations [3, 4]).

An apparent discrepancy has recently been noted, however [5, 6]), between the calculated and measured values of the optical potential for pions having energies of a few GeV. This discrepancy seems to indicate that, at 3 GeV, for instance, the imaginary part of the potential is about 20 % greater in magnitude than the calculated value.

It has long been known that the elementary theory of a refractive index (or optical model potential) require modifications which depend on the structure of the scattering medium [7]). Such corrections were estimated by Francis and Watson [8, 9]) to illustrate a general theory of the optical model potential. These estimates were qualitative because of the limited information then available concerning the pion-nucleon scattering amplitudes. Later studies of this effect have been made by Glauber [10]) and, in particular, by Beg [11]). These

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studies do not seem, however, to have emphasized sufficiently the sensitivity of the results to the nuclear model assumed. For example, the expressions obtained by Beg [11]) and Watson [9]) are quite different †. We shall now discuss this problem more generally and demonstrate explicitly the relations between the various models.

The results are summarized in sect. 2, with numerical values given in sect. 3. Relevant derivations are then presented in sect. 4. Our analysis will be restricted to high energy scattering, for which the pions have energies in the GeV range.

2. Description of the Potential

In this section we shall state the results to be obtained later in our discussion. The elastic interaction of a π^+ or π^- meson with a nucleus containing Z protons and $N = A-Z$ neutrons is represented by an optical model potential

$$v(\mathbf{x}) = \rho(\mathbf{x})V. \tag{2.1}$$

Here V is the "well depth" and $\rho(\mathbf{x})$ is the density of nucleons at a point $\mathbf{x}$ in the nucleus, normalized according to the condition that

$$\int \rho(\mathbf{x})\mathrm{d}^3x = \mathscr{V}_A \equiv \frac{4\pi}{3} R_A{}^3, \tag{2.2}$$

the nuclear volume. It is convenient to define also the density

$$P(\mathbf{x}) \equiv \frac{1}{\mathscr{V}_A}\rho(\mathbf{x}), \tag{2.3}$$

so that

$$\int P(\mathbf{x})\mathrm{d}^3x = 1. \tag{2.4}$$

The well depth V in eq. (2.1) is considered to be a function of the momentum k_0 of the *incident* meson. The Fourier transform of eq. (2.1) is therefore

$$\langle \mathbf{q}|v|\mathbf{q}_0\rangle = (2\pi)^{-3}\int \mathrm{e}^{-i(\mathbf{q}-\mathbf{q}_0)\cdot\mathbf{x}} v(\mathbf{x})\mathrm{d}^3x = c(|\mathbf{q}-\mathbf{q}_0|)\left[\frac{\mathscr{V}_A V}{(2\pi)^3}\right], \tag{2.5}$$

where

$$c(K) \equiv \int P(\mathbf{x})\mathrm{e}^{-i\mathbf{K}\cdot\mathbf{x}}\mathrm{d}^3x, \tag{2.6}$$

and V depends parametrically on k_0.

The Schrödinger equation which describes pion-nucleus scattering in the

† Beg erroneously states that the potential in ref. [9]) is incorrect. It is correct but differs from Beg's result in that a different nuclear model is used.

barycentric coordinate system is

$$[\varepsilon+v]\psi = \varepsilon_0\psi, \tag{2.7}$$

where $\varepsilon_0 = [\mu^2+k_0^2]^{\frac{1}{2}}$ is the incident energy and ε is the kinetic energy operator for the meson. [A kinematical correction for the finite rest-mass of the nucleus may be introduced into eq. (2.7) if necessary.]

We wish to express V in terms of the scattering matrices l_α ($\alpha = 1,2,\ldots,A$) for pion scattering by nucleon α. If the initial momenta of the nucleon and pion are $\mathbf{P}_{0\alpha}$ and $\mathbf{q}_0$, respectively, and if the scattering leads to the final momenta $\mathbf{P}_\alpha$ and $\mathbf{q}$, then

$$l_\alpha = \langle \mathbf{q}, \mathbf{P}_\alpha | t_\alpha | \mathbf{q}_0, \mathbf{P}_{0\alpha}\rangle \delta(\mathbf{q}+\mathbf{P}_\alpha-\mathbf{q}_0-\mathbf{P}_{0\alpha}). \tag{2.8}$$

Here t_α is defined only on the "momentum shell". When we wish to consider l_α and t_α as operating on the nucleon coordinates, but as functions of $\mathbf{q}$ and $\mathbf{q}_0$, we write

$$l_\alpha = l_\alpha(\mathbf{q}, \mathbf{q}_0), \qquad t_\alpha = t_\alpha(\mathbf{q}, \mathbf{q}_0). \tag{2.9}$$

It is convenient to project t_α onto the isobaric spin substates corresponding to $I = \frac{3}{2}$ and $I = \frac{1}{2}$. This may be effected with the respective projection operators $E_{\frac{3}{2}}$ and $E_{\frac{1}{2}}$, so that

$$t_\alpha = t_\alpha(\tfrac{3}{2})E_{\frac{3}{2}}+t_\alpha(\tfrac{1}{2})E_{\frac{1}{2}}. \tag{2.10}$$

We shall find that only the forward scattering amplitudes, for which $\mathbf{q} = \mathbf{q}_0 = \mathbf{k}_0$, enter into our (approximate) final expressions for V. The scattering matrices for forward scattering will be written as t_α^0. In particular, we have

$$t_\alpha^0(\tfrac{3}{2}) = t_0(\tfrac{3}{2}), \qquad t_\alpha^0(\tfrac{1}{2}) = t^0(\tfrac{1}{2}), \tag{2.11}$$

since these do not depend on the nucleon index α.

The *scattering amplitudes* for the $I = \frac{3}{2}$ and the $I = \frac{1}{2}$ states are, respectively,

$$f_{\frac{3}{2}} = -(2\pi)^2\varepsilon_0 t^0(\tfrac{3}{2}), \qquad f_{\frac{1}{2}} = -(2\pi)^2\varepsilon_0 t^0(\tfrac{1}{2}). \tag{2.12}$$

In terms of these we may define

$$f \equiv \tfrac{1}{3}[2f_{\frac{3}{2}}+f_{\frac{1}{2}}], \qquad f_\tau \equiv \tfrac{1}{3}[f_{\frac{3}{2}}-f_{\frac{1}{2}}]. \tag{2.13}$$

In addition to being scattered, a meson may be absorbed by a nucleus. It is understood that the effect of this is also included in the optical potential V. The effective cross section for absorption, per nucleon, is conventionally [12]) written as $\frac{1}{2}\Gamma\sigma_d$, where σ_d is the cross section for absorption on a deuteron and $\Gamma \approx 4$ (ref. [13])).

The mean free path for absorption of a meson in nuclear matter is then λ_a, where [2])

$$\frac{1}{\lambda_a} = \frac{1}{2}\frac{A}{\mathscr{V}_A}\Gamma\sigma_d. \tag{2.14}$$

[In adopting this expression, we are assuming that $Z = N$ in the nucleus. This is justified, since the contribution from (2.14) to V is both small and poorly known for pions in the GeV range.] Following the analysis of Cronin *et al* [5]) and of Beg [11]), we shall take

$$\sigma_d \approx 0.5 \times 10^{-27} \text{ cm}^2 \tag{2.15}$$

for pions in the 1—5 GeV energy range.

In the usual elementary theory of the optical potential, one expresses V in eq. (2.1) as

$$V_1(\pi^\pm, k_0) = -\frac{2\pi}{\varepsilon_0}\left(\frac{A}{\mathscr{V}_A}\right)\left[f \pm 2\frac{T_3}{A} f_\tau + i\frac{k_0 \Gamma \sigma_d}{8\pi}\right], \tag{2.16}$$

where $V_1(\pi^\pm, k_0)$ refers to π^+ or π^- mesons, respectively, and T_3 is the third component of the nuclear isobaric spin

$$T_3 = \tfrac{1}{2}(Z-N). \tag{2.17}$$

The expression (2.16) has been tabulated [1, 2]) using dispersion relations [3, 4]) to evaluate f and f_τ.

Corrections to eq. (2.16) depend in some detail on nuclear structure. The leading correction terms are also expressible in terms of f and f_τ and certain nuclear parameters which we shall call "correlation lengths."

To define these, we need consider the joint probability of finding two nucleons 1 and 2 at points $\mathbf{z}_1$ and $\mathbf{z}_2$. If the nucleons are in a space-symmetric state, this is written as

$$P_s(\mathbf{z}_1, \mathbf{z}_2) = P(\mathbf{z}_1)P(\mathbf{z}_2)[1+G_s(\mathbf{z}_1-\mathbf{z}_2)]. \tag{2.18}$$

For a space-antisymmetric state the corresponding probability is

$$P_a(\mathbf{z}_1, \mathbf{z}_2) = P(\mathbf{z}_1)P(\mathbf{z}_2)[1+G_a(\mathbf{z}_1-\mathbf{z}_2)]. \tag{2.19}$$

Here P is defined by eq. (2.3); G_s and G_a are the so-called "pair correlation functions." The probabilities P_s and P_a are considered to be normalized:

$$\int P_s(\mathbf{z}, \mathbf{z}')\mathrm{d}^3z\mathrm{d}^3z' = 1, \qquad \int P_a(\mathbf{z}, \mathbf{z}')\mathrm{d}^3z\mathrm{d}^3z' = 1. \tag{2.20}$$

The correlation lengths mentioned above, R_s and R_a, are defined as

$$R_s = \int_0^\infty G_s(r)\mathrm{d}r, \qquad R_a = \int_0^\infty G_a(r)\mathrm{d}r. \tag{2.21}$$

[A more precise definition, taking more careful account of the upper limit of integration and the finite nuclear size, is given in eq. (4.26).]

As will be shown in sect. 4, the well depth V in eq. (2.1) is approximately

given by the expression

$$V(\pi^{\pm}, k_0) = \left[-\frac{1}{A}\left(1+2\left(\frac{f_\tau}{f}\right)^2\right)\right][V_1+V_2]. \tag{2.22}$$

Here V_1 is given by eq. (2.16), and

$$V_2(\pi^{\pm}, k_0) = \frac{i(2\pi)^2}{2\varepsilon_0 k_0}\left(\frac{A}{\mathscr{V}_A}\right)^2\left\{-(R_s+R_a)\left[f^2\left(1-\frac{1}{A}\right)-\frac{2}{A}f_\tau^2 \pm 4\frac{T_3}{A}ff_\tau\right]\right.$$
$$\left.+(R_s-R_a)\left[\tfrac{1}{4}f^2\left(1-\frac{16}{A}\right)+\tfrac{1}{2}f_\tau^2 \pm 2\frac{T_3}{A}ff_\tau\right]\right\}. \tag{2.23}$$

In obtaining eq. (2.22) we have neglected corrections of relative order $1/A^2$ and further terms of order

$$\frac{1}{\varepsilon_0 k_0^2}\left(\frac{A}{\mathscr{V}_A}\right)^3 (R_s \pm R_a)^2 f^2.$$

On solving eq. (2.7) one obtains a differential scattering cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$. According to the exact theory of Francis and Watson [8]), there are additional corrections $O(1/A)$ required to obtain the correct cross section $(\mathrm{d}\sigma/\mathrm{d}\Omega)_c$. To lowest order in $1/A$, one finds

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_c = \left[1-\frac{1}{A}\left[1+2\left(\frac{f_\tau}{f}\right)^2\right]\right]^{-2}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}. \tag{2.24}$$

The expression (2.23) may also be evaluated from the dispersion relations. This will be described in the next section. The potential V, given by eq. (2.22), may then be used in eq. (2.7) to find $\mathrm{d}\sigma/\mathrm{d}\Omega$. Finally, comparison with experimental cross sections may then be made using the expression (2.24).

3. Evaluation of the Potential

The real parts of the amplitudes f and f_τ may be evaluated using the dispersion relations [3, 4]). The imaginary parts are given by the optical theorem in terms of pion-nucleon scattering cross sections. We find, then,

$$f = \tfrac{1}{2}(D^+ + D^-) + \frac{ik_0}{8\pi}(\sigma^+ + \sigma^-), \qquad f_\tau = \tfrac{1}{2}(D^+ - D^-) + \frac{ik_0}{8\pi}(\sigma^+ - \sigma^-). \tag{3.1}$$

The amplitudes D^+ and D^- have been defined in refs. [3, 4]). They are the real parts of the forward scattering amplitudes of $\pi^{\pm}$ mesons on protons in the laboratory coordinate system. The quantities σ^+ and σ^- are the corresponding total cross sections for π^+- or π^-- proton scattering [14]). The quantities D^+ and D^- have been evaluated [4]) only up to 2.6 GeV. Since they are small at

TABLE I

The parameters of the $\pi^{\pm}$-nucleus optical model potential as defined by eq

T_π (GeV)	V_{00}^{R} (MeV)	V_{01}^{R} (MeV)	V_{02}^{R} (MeV)	V_{00}^{I} (MeV)	V_{01}^{I} (MeV)	V_{02}^{I} (MeV)	V_{10}^{R} (MeV · fm^{-1})	V_{11}^{R} (MeV · fm^{-1})	V_{12}^{R} (MeV · fm^{-1})	V_{10}^{I} (MeV · fm^{-1})
1.0	11.9	−22.0	−25.5	−47.0	44.7	17.7	−2.9	4.0	7.2	5.1
1.1	7.8	−12.0	−23.7	−45.5	39.0	6.2	−1.8	2.2	5.7	4.9
1.2	6.2	−5.0	−22.5	−50.0	43.4	−4.0	−1.6	1.3	5.5	6.0
1.3	9.4	−7.8	−15.0	−50.5	47.0	−6.7	−2.4	2.1	3.5	6.0
1.4	13.5	−12.8	−6.4	−49.0	47.4	−5.6	−3.3	3.2	1.2	5.4
1.5	16.2	−16.0	−3.5	−46.5	44.8	−3.1	−3.8	3.7	0.6	4.6
1.6	15.6	−16.1	−6.0	−42.9	41.3	+2.4	−3.3	3.4	1.5	3.8
1.7	12.6	−13.1	−6.1	−41.7	40.1	3.1	−2.6	2.7	1.5	3.6
1.8	10.6	−11.1	−5.6	−41.0	39.5	3.2	−2.2	2.2	1.3	3.7
1.9	9.5	−10.0	−5.6	−40.5	39.0	3.3	−1.9	1.9	1.3	3.7
2.0	8.6	−9.1	−5.9	−40.3	38.7	2.7	−1.7	1.8	1.3	3.7
2.2	7.4	−7.7	−5.6	−40.5	38.8	1.6	−1.5	1.5	1.2	3.8
2.4	6.7	−6.8	−5.6	−40.8	39.1	0.4	−1.3	1.4	1.1	3.9
2.6	6.1	−6.1	−5.4	−41.0	39.2	0.0	−1.2	1.2	1.1	3.9
3.0	5.3	−5.3	−4.7	−41.0	39.4	0.0	−1.1	1.1	1.0	3.9
3.5	4.6	−4.5	−4.1	−41.0	39.5	0.0	−0.9	0.9	0.8	4.0
4.0	4.0	−4.0	−3.6	−41.1	39.6	0.0	−0.8	0.8	0.7	4.0
4.5	3.6	−3.6	−3.2	−41.1	39.6	0.0	−0.7	0.7	0.7	4.0
5.0	3.2	−3.2	−2.9	−41.1	39.7	0.0	−0.7	0.7	0.6	4.0

these high energies, we have extrapolated them as constant in the range 2.6–5 GeV.

Eq. (2.22) may now be evaluated. We write it (to order A^{-1}) in the form

$$V(\pi^{\pm}, k_0) = \frac{1}{\lambda^3}\left\{\left[V_{00}^{R}+\frac{1}{A}V_{01}^{R}\pm\frac{T_3}{A}V_{02}^{R}\right]+i\left[V_{00}^{I}+\frac{1}{A}V_{01}^{I}\pm\frac{T_3}{A}V_{02}^{I}\right]\right\}$$
$$+\frac{1}{\lambda^6}(R_s+R_a)\left\{\left[V_{10}^{R}+\frac{1}{A}V_{11}^{R}\pm\frac{T_3}{A}V_{12}^{R}\right]+i\left[V_{10}^{I}+\frac{1}{A}V_{11}^{I}\pm\frac{T_3}{A}V_{12}^{I}\right]\right\}$$
$$+\frac{1}{\lambda^6}(R_s-R_a)\left\{\left[V_{20}^{R}+\frac{1}{A}V_{21}^{R}\pm\frac{T_3}{A}V_{22}^{R}\right]+i\left[V_{20}^{I}+\frac{1}{A}V_{21}^{I}\pm\frac{T_3}{A}V_{22}^{I}\right]\right\}. \quad (3.2)$$

Here λ is related to the nuclear radius R_A [eq. (2.2)] by

$$R_A = 1.2\lambda A^{\frac{1}{3}}\ \text{fm}. \quad (3.3)$$

For a Fermi gas model of the nucleus we evaluate the quantities R_s and R_a as follows:

$$P_{s(a)}(\mathbf{x}, \mathbf{x}') = \tfrac{1}{2}\sum_{\alpha\neq\beta}\langle 0|\delta(\mathbf{x}-\mathbf{z}_\alpha)\delta(\mathbf{x}'-\mathbf{z}_\beta)(1\pm P_{\alpha\beta})|0\rangle, \quad (3.4)$$

where $P_{\alpha\beta}$ is the space-exchange operator (see eq. (4.13)). In terms of the single-

(3.2) ($T_\pi = \varepsilon_0 - \mu$ is the pion kinetic energy in the barycentric system)

V_{11}^{I} (MeV · fm^{-1})	V_{12}^{I} (MeV · fm^{-1})	V_{20}^{R} (MeV · fm^{-1})	V_{21}^{R} (MeV · fm^{-1})	V_{22}^{R} (MeV · fm^{-1})	V_{20}^{I} (MeV · fm^{-1})	V_{21}^{I} (MeV · fm^{-1})	V_{22}^{I} (MeV · fm^{-1})
−4.7	−2.7	1.0	−11.4	−3.6	−1.2	20.4	1.3
−4.3	−0.5	0.6	−7.2	−2.9	−1.1	19.7	0.2
−5.4	+1.7	0.3	−6.2	−2.8	−1.4	24.1	−0.9
−5.8	2.4	0.5	−9.6	−1.8	−1.5	24.1	−1.2
−5.4	1.8	0.8	−13.3	−0.6	−1.4	21.6	−0.9
−4.6	1.0	0.9	−15.0	−0.3	−1.1	18.3	−0.5
−3.8	−0.0	0.9	−13.4	−0.7	−0.9	15.2	+0.0
−3.7	−0.3	0.7	−10.4	−0.7	−0.9	10.4	0.1
−3.7	−0.4	0.6	−8.6	−0.7	−0.9	14.9	0.2
−3.7	−0.9	0.5	−7.6	−0.7	−0.9	14.8	0.5
−3.7	−0.3	0.4	−6.8	−0.6	−0.9	14.7	0.1
−3.7	−0.1	0.4	−6.0	−0.6	−0.9	15.0	0.0
−3.8	+0.1	0.3	−5.4	−0.6	−1.0	15.4	0.0
−3.9	+0.2	0.3	−4.9	−0.6	−1.0	15.6	0.0
−3.9	0.1	0.3	−4.2	−0.5	−1.0	15.7	0.0
−3.9	0.1	0.2	−3.7	−0.4	−1.0	15.8	0.0
−4.0	0.1	0.2	−3.2	−0.4	−1.0	15.9	0.0
−4.0	0.1	0.2	−2.9	−0.3	−1.0	15.9	0.0
−4.0	0.1	0.2	−2.6	−0.3	−1.0	15.9	0.0

particle states

$$\varphi_l(1) = \frac{1}{\mathscr{V}_A^{\frac{1}{2}}} e^{i\mathbf{k}_l \cdot \mathbf{x}_1} i_l(1) s_l(1), \tag{3.5}$$

where i_l and s_l are the isobaric spin and spin eigenfunctions, respectively, we have

$$P_{s(a)} = \tfrac{1}{2} \sum_{l \neq m} \{\langle \varphi_l(1)\varphi_m(2)|(1 \pm P_{lm})\delta(\mathbf{x}-\mathbf{z}_1)\delta(\mathbf{x}'-\mathbf{z}_2)|\varphi_l(1)\varphi_m(2)\rangle$$
$$-\langle \varphi_l(1)\varphi_m(2)|(1 \pm P_{lm})\delta(\mathbf{x}-\mathbf{z}_1)\delta(\mathbf{x}'-\mathbf{z}_2)|\varphi_l(2)\varphi_m(1)\rangle\}. \tag{3.6}$$

Defining

$$\bar{G}_s = [\tfrac{1}{4}A(\tfrac{1}{4}A-1)]^{-1} \sum_{\mathbf{k}_l \neq \mathbf{k}_m} e^{-i(\mathbf{k}_l - \mathbf{k}_m)\cdot(\mathbf{x}-\mathbf{x}')}, \tag{3.7}$$

we obtain

$$P_s = \frac{1}{\mathscr{V}_A^{\,2}}\left[1 + \frac{1-4/A}{1+4/A}\bar{G}_s\right], \qquad P_a = \frac{1}{\mathscr{V}_A^{\,2}}(1-\bar{G}_s), \tag{3.8}$$

$$\bar{G}_s \approx \frac{9\pi}{2} \frac{[J_{\frac{3}{2}}(k_F r)]^2}{(k_F r)^3} \tag{3.9}$$

where $r = |\mathbf{x}'-\mathbf{x}|$ and k_F is the Fermi momentum. Then [15]), one has

$$R_s + R_a = \frac{-(8/A)}{1+4/A}\frac{3\pi}{5k_F}, \qquad R_s - R_a = \frac{2}{1+4/A}\frac{3\pi}{5k_F}, \tag{3.10}$$

so that eq. (2.23) becomes

$$V_2(\pi^{\pm}, k_0) = i\frac{2\pi}{\varepsilon_0 k_0}\frac{k_F^2}{5}\left(\frac{A}{\mathscr{V}_A}\right)\left(f^2+2f_\tau^2\pm\frac{8T_3}{A}ff_\tau\right)\left(1-\frac{4}{A}\right)\frac{1}{\lambda^5}, \quad (3.11)$$

agreeing with the result of Beg [11]).

The result of Watson [9]) follows directly from the nuclear model in which one sets $R_s = R_a$. This assumes that the correlations are similar in spatially symmetric and antisymmetric states.

Brueckner and Gammel [16]) describe a wave function of the relative coordinate of a pair of nucleons in nuclear matter. Using their result we obtain

$$R_s \approx R_a = -0.84 \text{ fm}, \quad (3.12)$$

describing the correlations due to the " hard cores" of the nucleon-nucleon interaction. Some indication of a similar result has recently [17]) been deduced from experiment †.

The optical potentials for these two nuclear models (neglecting the $1/A$ corrections) are presented in table 2, where we have written V^F and V^B for

TABLE 2

The well-depths of the $\pi^{\pm}$-nucleus optical model potential for a Fermi gas model of the nucleus (V^F = Re V^F+i Im V^F) and a Brueckner model of the nucleus (V^B = Re V^B + i Im V^B) (T_π is the barycentric kinetic energy of the pion. We have neglected the $1/A$ corrections)

T_π(GeV)	Re V^F(MeV)	Im V^F(MeV)	Re V^B(MeV)	Im V^B(MeV)
1.0	14.9	−49.2	16.6	−54.3
1.1	9.4	−47.3	10.8	−52.4
1.2	7.2	−52.6	8.8	−58.7
1.3	11.0	−53.4	13.4	−59.2
1.4	15.9	−51.7	19.1	−56.7
1.5	19.0	−48.5	22.5	−52.8
1.6	18.2	−44.3	21.2	−47.9
1.7	14.6	−43.1	17.0	−46.4
1.8	12.3	−42.4	14.3	−45.9
1.9	10.9	−41.9	12.6	−45.3
2.0	9.9	−41.6	11.5	−45.1
2.2	8.6	−41.9	9.9	−45.4
2.4	7.7	−42.3	8.9	−46.0
2.6	7.0	−42.5	8.1	−46.1
3.0	6.1	−42.6	7.1	−46.2
3.5	5.2	−42.6	6.1	−46.3
4.0	4.6	−42.7	5.4	−46.4
4.5	4.1	−42.7	4.8	−46.4
5.0	3.7	−42.7	4.3	−46.4

the Fermi gas model and the Brueckner model, respectively, in the form

$$V_F(\pi^{\pm}, k_0) = \text{Re } V^F + i \text{ Im } V^F, \qquad V_B(\pi^{\pm}, k_0) = \text{Re } V^B + i \text{ Im } V^B, \quad (3.13)$$

and we have assumed $\lambda = 1$.

† Ref. [17]) describes possible experimental techniques for determining G_s and G_a.

4. Derivation of the Second-Order Potential

The optical model potential $\langle \mathbf{q}|\mathscr{V}|\mathbf{q}_0\rangle$, defined between momentum states $\mathbf{q}$ and $\mathbf{q}_0$ of the pion †, may be expressed as a series of terms:

$$\mathscr{V} = \mathscr{V}_1+\mathscr{V}_2+\ldots+\mathscr{V}_a. \tag{4.1}$$

The last term here represents the effects of pion absorption in the nucleus, as discussed in connection with eq. (2.14), and is contained in eq. (2.16). According to Francis and Watson [appendix to ref. [8])], the scattering matrix T_c for pion-nucleus scattering is defined by the integral equation

$$T_c = \mathscr{V}+\mathscr{V}\frac{\left(1-\frac{1}{A}\right)}{\varepsilon_0+i\eta-\varepsilon}T_c. \tag{4.2}$$

The scattering cross section is then

$$\left(\frac{d\sigma}{d\Omega}\right)_c = (2\pi)^4\varepsilon_0^2|T_c|^2. \tag{4.3}$$

The first term in eq. (4.1) is given by the familiar expression

$$\langle \mathbf{q}|\mathscr{V}_1|\mathbf{q}_0\rangle = \sum_{\alpha=1}^{A}\langle 0|\hat{t}_\alpha(\mathbf{q},\mathbf{q}_0)|0\rangle, \tag{4.4}$$

where $|0\rangle$ represents the initial, ground state nuclear wave function. The notation introduced in eq. (2.9) has been used here.

The second term in eq. (4.1) is

$$\langle \mathbf{q}|\mathscr{V}_2|\mathbf{q}_0\rangle = \sum_{\substack{\alpha,\beta=1\\(\alpha\neq\beta)}}^{A}\sum_{\gamma(\neq 0)}\int \frac{d^3q'\langle 0|\hat{t}_\alpha(\mathbf{q},\mathbf{q}')|\gamma\rangle\langle\gamma|\hat{t}_\beta(\mathbf{q}',\mathbf{q}_0)|0\rangle}{\varepsilon_0+i\eta+W_0-\varepsilon_{q'}-W_\gamma}. \tag{4.5}$$

Here $|\gamma\rangle$ represents the wave function for a particular nuclear state γ, for which W_γ is the energy. The sum over γ in eq. (4.5) runs over all states except $|0\rangle$, as indicated.

We shall first simplify eq. (4.5) by neglecting the energy difference W_0-W_γ in the denominator. For sufficiently high energies this is justified, since the error incurred is of the order [17, 19]) of $\frac{1}{3}(K_{av}/\varepsilon_0)$, where $K_{av}\approx 30$ MeV is the average kinetic energy of a nucleon bound in the nucleus. For multi-GeV pions, this error appears quite negligible. Eq. (4.5) then reduces to

$$\langle \mathbf{q}|\mathscr{V}_2|\mathbf{q}_0\rangle = \sum_{\alpha\neq\beta=1}^{A}\int\frac{d^3q'}{\varepsilon_0+i\eta-\varepsilon_{q'}}$$
$$\{\langle 0|\hat{t}_\alpha(\mathbf{q},\mathbf{q}')\hat{t}_\beta(\mathbf{q}',\mathbf{q}_0)|0\rangle-\langle 0|\hat{t}_\alpha(\mathbf{q},\mathbf{q}')|0\rangle\langle 0|\hat{t}_\beta(\mathbf{q}',\mathbf{q}_0)|0\rangle\}. \tag{4.6}$$

† The relation of $\mathscr{V}$ to the v of eq. (2.1) is indicated below.

We now set †

$$t_\alpha(\mathbf{q}, \mathbf{q}_0)|0\rangle = e^{-i(\mathbf{q}-\mathbf{q}_0)\cdot\mathbf{z}_\alpha} t_\alpha(\mathbf{q}, \mathbf{q}_0)|0\rangle, \tag{4.7}$$

where $\mathbf{z}_\alpha$ is the coordinate of nucleon α and t_α is given by eq. (2.8). We assume that, because of the high energy of the pion, we may set [17]) $\mathbf{P}_{0\alpha} = 0$ in t_α. Thus t_α depends on only $\mathbf{q}$, $\mathbf{q}_0$ and the spin and isobaric spin matrices $\boldsymbol{\sigma}_\alpha$ and $\boldsymbol{\tau}_\alpha$.

With the use of (4.7), eq. (4.4) becomes

$$\langle \mathbf{q}|\mathscr{V}_1|\mathbf{q}_0\rangle = \sum_{\alpha=1}^{A} \langle 0|e^{-i(\mathbf{q}-\mathbf{q}_0)\cdot\mathbf{z}_\alpha} t_\alpha|0\rangle. \tag{4.8}$$

This expression becomes quite small for $|\mathbf{q}-\mathbf{q}_0| \gg \hbar/R_A$, where R_A is the nuclear radius. As is customary, we assume t_α to vary by a negligible amount over the relevant range of $\mathbf{q}$ and $\mathbf{q}_0$ and set $\mathbf{q} = \mathbf{q}_0 = \mathbf{k}_0$ in t_α, so that $t_\alpha = t_\alpha^0$. Then there results

$$\begin{aligned}\langle \mathbf{x}|\mathscr{V}_1|\mathbf{x}'\rangle &= (2\pi)^{-3} \int d^3q d^3q_0 e^{i\mathbf{q}\cdot\mathbf{x}} e^{-i\mathbf{q}_0\cdot\mathbf{x}'} \langle \mathbf{q}|\mathscr{V}_1|\mathbf{q}_0\rangle \\ &= (2\pi)^3 \sum_\alpha \langle 0|\delta(\mathbf{x}-\mathbf{z}_\alpha)\delta(\mathbf{x}'-\mathbf{z}_\alpha) t_\alpha^0|0\rangle = \delta(\mathbf{x}-\mathbf{x}') v_1(x), \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} v_1(x) &= \sum_\alpha (2\pi)^3 \langle 0|\delta(\mathbf{x}-\mathbf{z}_\alpha) t_\alpha^0|0\rangle \\ &= (2\pi)^3 A t_{\mathrm{av}} \langle 0|\delta(\mathbf{x}-\mathbf{z}_\alpha)|0\rangle = \rho(x) V_1. \end{aligned} \tag{4.10}$$

We have written t_{av} as the average of t_α^0 for the neutrons and protons in the nucleus. This gives the first-order potential (2.16). [We then included the term $\mathscr{V}_3$ in the last line of eq. (4.10).]

To evaluate the second-order potential $\mathscr{V}_2$, we use the relation (4.7) in eq. (4.6) to obtain

$$\begin{aligned}\langle \mathbf{q}|\mathscr{V}_2|\mathbf{q}_0\rangle = \sum_{\alpha\neq\beta} \int \frac{d^3q'}{\varepsilon_0 + i\eta - \varepsilon_{q'}} \{ &\langle 0|e^{-i(\mathbf{K}'\cdot\mathbf{z}_\alpha + \mathbf{K}\cdot\mathbf{z}_\beta)} t_\alpha t_\beta|0\rangle \\ &- \langle 0|e^{-i\mathbf{K}'\cdot\mathbf{z}_\alpha} t_\alpha|0\rangle \langle 0|e^{-i\mathbf{K}\cdot\mathbf{z}_\beta} t_\beta|0\rangle \}, \end{aligned} \tag{4.11}$$

where

$$\mathbf{K} \equiv \mathbf{q}'-\mathbf{q}_0, \qquad \mathbf{K}' \equiv \mathbf{q}-\mathbf{q}'. \tag{4.12}$$

We now make use of a technique due to Lax and Feshbach [20]) for the evaluation of matrix elements such as those occurring in eq. (4.11). [The specific evaluation is almost identical with that given by Fowler and Watson [17]), and will therefore not be described in detail.] Let $P_{\alpha\beta}$ be the space exchange operator which interchanges $\mathbf{z}_\alpha$ and $\mathbf{z}_\beta$ and write

$$|\mathrm{s}\rangle = \tfrac{1}{2}[1+P_{\alpha\beta}]|0\rangle, \qquad |\mathrm{a}\rangle = \tfrac{1}{2}[1-P_{\alpha\beta}]|0\rangle. \tag{4.13}$$

† See for example, eq. (3.5) of ref. [17]).

These represent the parts of $|0\rangle$ which are symmetric and antisymmetric, respectively, in the variables $\mathbf{z}_\alpha$ and $\mathbf{z}_\beta$. Therefore, if $\mathcal{O}_{\alpha\beta}$ is an operator symmetric in α and β, we have

$$\langle 0|\mathcal{O}_{\alpha\beta}|0\rangle = \langle \mathrm{s}|\mathcal{O}_{\alpha\beta}|\mathrm{s}\rangle + \langle \mathrm{a}|\mathcal{O}_{\alpha\beta}|\mathrm{a}\rangle. \tag{4.14}$$

We assume that there is a single probability distribution $P_\mathrm{s}(\mathbf{z}_\alpha, \mathbf{z}_\beta)$ for nucleons in space symmetric states and a similar distribution $P_\mathrm{a}(\mathbf{z}_\alpha, \mathbf{z}_\beta)$ for space antisymmetric states †. These may evidently be written in the general form

$$\begin{aligned} P_\mathrm{s}(\mathbf{z}_\alpha, \mathbf{z}_\beta) &= P(\mathbf{z}_\beta)[P(\mathbf{z}_\alpha) + Q_\mathrm{s}(\mathbf{z}_\alpha - \mathbf{z}_\beta, \mathbf{z}_\beta)], \\ P_\mathrm{a}(\mathbf{z}_\alpha, \mathbf{z}_\beta) &= P(\mathbf{z}_\beta)[P(\mathbf{z}_\alpha) + Q_\mathrm{a}(\mathbf{z}_\alpha - \mathbf{z}_\beta, \mathbf{z}_\beta)]. \end{aligned} \tag{4.15}$$

The functions Q_s and Q_a represent conditional probabilities. The expressions (4.15) are generalizations of those given by eqs. (2.18) and (2.19). For a sufficiently large nucleus, we would anticipate being able to set

$$Q_\mathrm{s}(\mathbf{z}_\alpha - \mathbf{z}_\beta, \mathbf{z}_\beta) = P(\mathbf{z}_\alpha) G_\mathrm{s}(\mathbf{z}_\alpha - \mathbf{z}_\beta). \tag{4.16}$$

We may now identify the operator $\mathcal{O}_{\alpha\beta}$ in eq. (4.14) with the expression

$$\mathrm{e}^{-i(\mathbf{K}'\cdot\mathbf{z}_\alpha + \mathbf{K}\cdot\mathbf{z}_\beta)} t_\alpha t_\beta$$

in eq. (4.11). In evaluating eq. (4.11) we must then consider terms such as

$$\langle s|\mathrm{e}^{-i(\mathbf{K}'\cdot\mathbf{z}_\alpha + \mathbf{K}\cdot\mathbf{z}_\beta)} t_\alpha t_\beta|\mathrm{s}\rangle = \left\{\int P_\mathrm{s}(\mathbf{z}_\alpha, \mathbf{z}_\beta)\mathrm{e}^{-i(\mathbf{K}'\cdot\mathbf{z}_\alpha + \mathbf{K}\cdot\mathbf{z}_\beta)}\mathrm{d}^3 z_\alpha \mathrm{d}^3 z_\beta\right\}\langle s|t_\alpha t_\beta|s\rangle, \tag{4.17}$$

since t_α and t_β depend on spin and isobaric spin variables only.
Now, we have

$$\int \frac{\mathrm{d}^3 q'}{\varepsilon_0 + i\eta - \varepsilon_{q'}} \int \mathrm{d}^3 z_\alpha \mathrm{e}^{-i\mathbf{K}'\cdot\mathbf{z}_\alpha} P(\mathbf{z}_\alpha) \int \mathrm{d}^3 z_\beta \mathrm{e}^{-i\mathbf{K}\cdot\mathbf{z}_\beta} P(\mathbf{z}_\beta) \langle s|t_\alpha t_\beta|s\rangle \approx B\langle s|t_\alpha^0 t_\beta^0|s\rangle, \tag{4.18}$$

where

$$B \equiv \int \frac{\mathrm{d}^3 q'}{\varepsilon_0 + i\eta - \varepsilon_{q'}} c(K')c(K), \tag{4.19}$$

and $c(K)$ was defined by eq. (2.6). We have here set $t_\alpha = t_\alpha^0$, $t_\beta = t_\beta^0$ for reasons discussed in connection with eq. (4.9).

It is convenient to introduce

$$J_\mathrm{s} \equiv \sum_{\alpha \neq \beta} \int \frac{\mathrm{d}^3 q' \mathrm{e}^{-i(\mathbf{K}'\cdot\mathbf{z}_\alpha + \mathbf{K}\cdot\mathbf{z}_\beta)}}{\varepsilon_0 + i\eta - \varepsilon_{q'}} \mathrm{d}^3 z_\alpha \mathrm{d}^3 z_\beta \times P(\mathbf{z}_\beta) Q_\mathrm{s}(\mathbf{z}_\alpha - \mathbf{z}_\beta, \mathbf{z}_\beta)\langle s|t_\alpha t_\beta|s\rangle, \tag{4.20}$$

and a similar quantity J_a for the space antisymmetric states. Then, eq. (4.11)

† The method used may be easily generalized to include separate probability distributions for each of the four states of two nucleons.

may be re-written as

$$\langle \mathbf{q}|\mathscr{V}_2|\mathbf{q}_0\rangle = B\{\sum_{\alpha\neq\beta}[\langle 0|t_\alpha{}^0 t_\beta{}^0|0\rangle - \langle 0|t_\alpha{}^0|0\rangle\langle 0|t_\beta{}^0|0\rangle]\} + J_s + J_a. \quad (4.21)$$

As is shown in the appendix, one has

$$\sum_{\alpha\neq\beta}[\langle 0|t_\alpha{}^0 t_\beta{}^0|0\rangle - \langle 0|t_\alpha{}^0|0\rangle\langle 0|t_\beta{}^0|0\rangle] = -2A\left[\frac{f_\tau}{(2\pi)^2\varepsilon_0}\right]^2 \equiv \Delta. \quad (4.22)$$

To simplify J_s and J_a, we take the Fourier transforms

$$\int \frac{\mathrm{d}^3q\,\mathrm{d}^3q_0}{(2\pi)^3}\,\mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}}\mathrm{e}^{-i\mathbf{q}_0\cdot\mathbf{x}'}\langle \mathbf{q}|J_s|\mathbf{q}_0\rangle \approx \delta(\mathbf{x}-\mathbf{x}')J_s(\mathbf{x}) \quad (4.23)$$

(with a similar equation for J_a), where

$$J_s(\mathbf{x}) = (2\pi)^3 P(\mathbf{x})\sum_{\alpha\neq\beta}\int \frac{\mathrm{d}^3q'\,\mathrm{d}^3R\,\mathrm{e}^{i\mathbf{K}_0\cdot\mathbf{R}}}{\varepsilon_0+i\eta-\varepsilon_{q'}} Q_s(\mathbf{R},\mathbf{x})\times\langle s|t_\alpha(\mathbf{k}_0,\mathbf{q}')t_\beta(\mathbf{q}',\mathbf{k}_0)|s\rangle. \quad (4.24)$$

Here we used $\mathbf{R} = \mathbf{z}_\alpha - \mathbf{z}_\beta$ and $\mathbf{K}_0 = \mathbf{q}' - \mathbf{k}_0$. [In deriving these results, we have assumed that $|\mathbf{q}-\mathbf{k}_0|R$ is small.] The quantity $J_a(\mathbf{x})$ is defined in a similar manner.

In the limit of very high energies, eq. (4.24) may be simply evaluated. We obtain

$$\begin{aligned} J_s(\mathbf{x}) &= P(\mathbf{x})\left[\frac{(2\pi)^6\varepsilon_0}{ik_0}\frac{R_s}{\mathscr{V}_A}\right]\sum_{\alpha\neq\beta}\langle s|t_\alpha{}^0 t_\beta{}^0|s\rangle, \\ J_a(\mathbf{x}) &= P(\mathbf{x})\left[\frac{(2\pi)^6\varepsilon_0}{ik_0}\frac{R_a}{\mathscr{V}_A}\right]\sum_{\alpha\neq\beta}\langle a|t_\alpha{}^0 t_\beta{}^0|a\rangle. \end{aligned} \quad (4.25)$$

Here we have introduced "correlation lengths" R_s and R_a, defined by

$$\frac{1}{\mathscr{V}_A}R_s = \int_0^\infty \mathrm{d}R Q_s(R\hat{k}_0,\mathbf{x}), \qquad \frac{1}{\mathscr{V}_A}R_a = \int_0^\infty \mathrm{d}R Q_a(R\hat{k}_0,\mathbf{x}). \quad (4.26)$$

Except when A is small, it is both convenient and reasonable to consider R_s and R_a to be independent of $\mathbf{x}$. In this case, eqs. (2.21) may be used to define R_s and R_a.

The "limit of very high energies" mentioned above may be seen to imply that $k_0R_s \gg 1$ and $k_0R_a \gg 1$. It is only because of the approximate evaluation given in eqs. (4.25) that we have restricted ourselves to quite high energy pions.

Then, on using eq. (4.13), we find

$$\begin{aligned} v_2(x) &\equiv J_s(x)+J_a(x) \\ &= P(x)\left[\frac{(2\pi)^6\varepsilon_0}{ik_0\mathscr{V}_A}\right]\sum_{\alpha\neq\beta}\{\tfrac{1}{2}(R_s+R_a)\langle 0|t_\alpha{}^0 t_\beta{}^0|0\rangle + \tfrac{1}{2}(R_s-R_a)\langle 0|t_\alpha{}^0 t_\beta{}^0 P_{\alpha\beta}|0\rangle\}. \end{aligned} \quad (4.27)$$

The expectation values are readily evaluated (see the appendix) to give

$$v_2(x) = \rho(x)\left[\frac{i2\pi^2}{k_0\varepsilon_0}\left(\frac{A}{\mathscr{V}_A}\right)^2\right]\left\{-(R_s+R_a)\left[f^2\left(1-\frac{1}{A}\right)-\frac{2}{A}f_\tau^2\pm 4\frac{T_3}{A}ff_\tau\right]\right.$$
$$\left.+(R_s-R_a)\left[\frac{f^2}{4}\left(1-\frac{16}{A}\right)+\tfrac{1}{2}f_\tau^2\pm 2\frac{T_3}{A}ff_\tau\right]\right\} \equiv \rho(x)V_2(\pi^\pm, k_0), \quad (4.28)$$

the last step being valid if R_s and R_a are considered to be independent of $\mathbf{x}$. Here V_2 is just the quantity given in eq. (2.23). Transformation of eq. (4.28) back to momentum space leads to

$$\langle \mathbf{q}|v_2|\mathbf{q}_0\rangle = c(|\mathbf{q}-\mathbf{q}_0|)\left[\frac{\mathscr{V}_A}{(2\pi)^3}V_2\right], \quad (4.29)$$

where the function c was defined by eq. (2.6).

Eqs. (4.22) and (4.29) may be substituted into (4.21) to give

$$\langle \mathbf{q}|\mathscr{V}_2|\mathbf{q}_0\rangle = B\Delta + c(|\mathbf{q}-\mathbf{q}_0|)\left[\frac{\mathscr{V}_A}{(2\pi)^3}V_2\right], \quad (4.30)$$

while eq. (4.1) becomes (valid to our approximation)

$$\langle \mathbf{q}|\mathscr{V}|\mathbf{q}_0\rangle = c(|\mathbf{q}-\mathbf{q}_0|)\left[\frac{\mathscr{V}_A}{(2\pi)^3}\right][V_1+V_2]+B\Delta. \quad (4.31)$$

The term involving B is cumbersome to use. Since it is $O(1/A)$, it may be discarded for applications to large nuclei. It may also be transformed into a more convenient form: to see this, let us introduce eq. (4.31) into eq. (4.2) and iterate, to obtain

$$T_c = c\frac{\mathscr{V}_A}{(2\pi)^3}[V_1+V_2]+B\left[\Delta+\left(1-\frac{1}{A}\right)\left(\frac{\mathscr{V}_A}{(2\pi)^3}V_1\right)^2\right]+\ldots. \quad (4.32)$$

Now, we attempt to obtain the same result by defining a new potential

$$v = c\frac{\mathscr{V}_A}{(2\pi)^3}\alpha[V_1+V_2], \quad (4.33a)$$

where α is a parameter to be determined, a new T-matrix

$$T = v+v\frac{1}{\varepsilon_0+i\eta-\varepsilon}T, \quad (4.33b)$$

and a relation

$$T_c = \beta T, \quad (4.33c)$$

where β is a second parameter. In order that eqs. (4.33c) and (4.32) be con-

sistent, we must have

$$\beta = \alpha^{-1}. \tag{4.34}$$

To first order in $1/A$ and in the potential strengths, we then find that

$$\alpha = 1 - \frac{1}{A}\left[1+2\left(\frac{f_\tau}{f}\right)^2\right]. \tag{4.35}$$

We are led immediately to eqs. (2.22) and (2.24).

Appendix

By the hypothesis of charge-independence the scattering matrices t_α^0 must be invariant under rotations in isospin space; that is, t_α^0 must have isospin dependence only of the form $\mathbf{I}\cdot\mathbf{I}$, where $\mathbf{I}$ is the total isospin of the pion-nucleon system. We introduce [18]) creation and annihilation operators $U_i^\dagger$, U_j $(i, j = 1, 2, 3)$ for the three types of pions $\omega_1, \omega_2, \omega_3$,

$$\mathbf{U} = U_1\hat{e}_1 + U_2\hat{e}_2 + U_3\hat{e}_3, \qquad [U_i, U_j^\dagger] = \delta_{ij}. \tag{A.1}$$

It is easy to show that the quantity

$$\mathbf{L} = i\mathbf{U}\times\mathbf{U}^\dagger \tag{A.2}$$

transforms like a vector in isospin space, and we therefore choose

$$\mathbf{I} = \tfrac{1}{2}\boldsymbol{\tau}_\alpha + i\mathbf{U}\times\mathbf{U}^\dagger, \tag{A.3}$$

where $\boldsymbol{\tau}_\alpha$ is the isospin operator of the α^{th} nucleon.

The most general form of t_α^0 is then

$$t_\alpha^0 = A_0\mathbf{U}^\dagger\cdot\mathbf{U} + iB_0\mathbf{U}^\dagger\times\mathbf{U}\cdot\boldsymbol{\tau}_\alpha, \tag{A.4}$$

where A_0 and B_0 are independent of the nucleon spin for forward scattering. One can verify the projection operators

$$E_{\frac{3}{2}} = \tfrac{1}{3}(2\mathbf{U}^\dagger\cdot\mathbf{U} - i\mathbf{U}^\dagger\times\mathbf{U}\cdot\boldsymbol{\tau}), \qquad E_{\frac{1}{2}} = \tfrac{1}{3}(\mathbf{U}^\dagger\cdot\mathbf{U} + i\mathbf{U}^\dagger\times\mathbf{U}\cdot\boldsymbol{\tau}), \tag{A.5}$$

allowing the identification of A_0 and B_0 with the two independent scattering matrices

$$t_\alpha^0 = (A_0 - B_0)E_{\frac{3}{2}} + (A_0 + 2B_0)E_{\frac{1}{2}} = t_\alpha^0(\tfrac{3}{2})E_{\frac{3}{2}} + t_\alpha^0(\tfrac{1}{2})E_{\frac{1}{2}}. \tag{A.6}$$

Since we assume only one pion to be present in the intermediate state, we find

$$t_\alpha^0 t_\beta^0 = A_0^2\mathbf{U}^\dagger\cdot\mathbf{U} + iA_0B_0\mathbf{U}^\dagger \times\mathbf{U}\cdot(\boldsymbol{\tau}_\alpha+\boldsymbol{\tau}_\beta) - B_0^2[(\boldsymbol{\tau}_\beta\cdot\mathbf{U}^\dagger)(\boldsymbol{\tau}_\alpha\cdot\mathbf{U}) - \boldsymbol{\tau}_\alpha\cdot\boldsymbol{\tau}_\beta\mathbf{U}^\dagger\cdot\mathbf{U}]. \tag{A.7}$$

Using the results of ref [17]), appendix B, and keeping terms of order A only,

we find (for unpolarized nuclei)

$$\langle 0|\sum_{\alpha\neq\beta} t_\alpha{}^0 t_\beta{}^0|0\rangle = \mathbf{U}^\dagger\cdot\mathbf{U}[A(A-1)A_0{}^2-2AB_0{}^2]+4iAA_0B_0\mathbf{U}^\dagger\times\mathbf{U}\cdot\hat{e}_3T_3,$$

$$\langle 0|\sum_{\alpha\neq\beta} t_\alpha{}^0 t_\beta{}^0 P_{\alpha\beta}|0\rangle$$

$$= \mathbf{U}^\dagger\cdot\mathbf{U}[-\tfrac{1}{4}A(A-16)A_0{}^2-\tfrac{1}{2}A^2B_0{}^2]-2iAA_0B_0\mathbf{U}^\dagger\times\mathbf{U}\cdot\hat{e}_3T_3, \tag{A.8}$$

$$\langle 0|\sum_{\alpha} t_\alpha{}^0|0\rangle = AA_0\mathbf{U}^\dagger\cdot\mathbf{U}+2iB_0\mathbf{U}^\dagger\times\mathbf{U}\cdot\hat{e}_3T_3.$$

To evaluate these expressions for incident charged pions, we introduce [18]) the pion state vectors

$$\omega^\pm = \mp\frac{1}{\sqrt{2}}(\omega_1\pm i\omega_2). \tag{A.9}$$

Then, using (A.6)

$$\langle 0|\sum_{\alpha\neq\beta} t_\alpha{}^0 t_\beta{}^0|0\rangle = \frac{A^2}{(2\pi)^4\varepsilon_0{}^2}\left[f^2\left(1-\frac{1}{A}\right)-\frac{2}{A}f_\tau{}^2\pm\frac{4T_3}{A}ff_\tau\right],$$

$$\langle 0|\sum_{\alpha\neq\beta} t_\alpha{}^0 t_\beta{}^0 P_{\alpha\beta}|0\rangle = \frac{A^2}{(2\pi)^4\varepsilon_0{}^2}\left[-\tfrac{1}{4}f^2\left(1-\frac{16}{A}\right)-\tfrac{1}{2}f_\tau{}^2\mp\frac{2T_3}{A}ff_\tau\right], \tag{A.10}$$

$$\langle 0|\sum_{\alpha} t_\alpha{}^0|0\rangle = \frac{-A}{(2\pi)^2\varepsilon_0}\left[f\pm\frac{2T_3}{A}f_\tau\right],$$

where the upper (lower) sign refers to incident positive (negative) pions, and f and f_τ are defined by eq. (2.13).

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A SURVEY OF GAMMA RAY BRANCHING RATIOS

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Abstract: Experimental gamma ray branching ratios are compared with the predictions of the single particle model. When both transitions are predicted to be predominantly E1 or M1 the experimental branching ratios show a distribution which is centred close to the predicted value with 69-76% of the transition pairs within a factor of 10 and 92-94% within a factor of 100 of this. When one transition is predicted to be E2 and the other M1, the median E2-M1 branching ratio is 63 times the predicted value with a spread similar to that of the E1-E1 and M1-M1 distributions.

The transition probability for the emission of a gamma ray of specified multipolarity and energy from a nucleus of specified mass number has been calculated by Weisskopf [1]) and by Moszkowski [2]) using a single-particle model. This estimate yields results which, in view of the drastically simplifying assumptions employed, are expected to be of qualitative significance only, often involving errors of several factors of ten. In a survey of gamma ray transitions in light nuclei ($A \leqq 20$) Wilkinson [3]) has shown that the single-particle model may be used with some consistency to predict the transition probability of E1 and M1 radiation. In particular he finds transition probabilities of E1 radiation to be centred about a value equal to 0.032 Weiskopf units with 85 % of the cases lying within a factor of seven of this, while for M1 radiation the corresponding values are 0.15 Weisskopf units with a spread of a factor of 20.

We report here the results of a somewhat different type of survey in which we consider only cases in which two or more transitions take place from the same level and compare the experimental branching ratios with those predicted by the single-particle model [2]). In this way we hope to see to what extent it predicts the *relative* transition probabilities of transitions starting at the same state. We make no attempt to examine the validity of the model. The purpose of the study is to determine to what extent the model may be used as a purely empirical tool in the assignment of reasonable values of spin and parity to levels involved in gamma ray transitions where the pertinent branching ratios have been measured.

The data employed have been taken from the nuclear data sheets prepared under the auspices of the National Academy of Sciences and the National

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Research Council [4]). We have taken all cases shown to date in which two or more gamma rays proceed from a level of known, or tentatively assigned, spin and parity to other levels of known, or tentatively assigned, spin and parity, and in which relative intensities have been measured. No distinction between light and heavy nuclei has been made; transitions in nuclei ranging in mass number from 16 to 209 have all been treated together. The statistical factor [2]) S has been ignored.

In fig. 1 is shown the distribution in the quantity

$$\log R \equiv \log \frac{\text{Experimental branching ratio}}{\text{Theoretical branching ratio}}$$

for cases in which both transitions are predicted † to be E1, when the branching

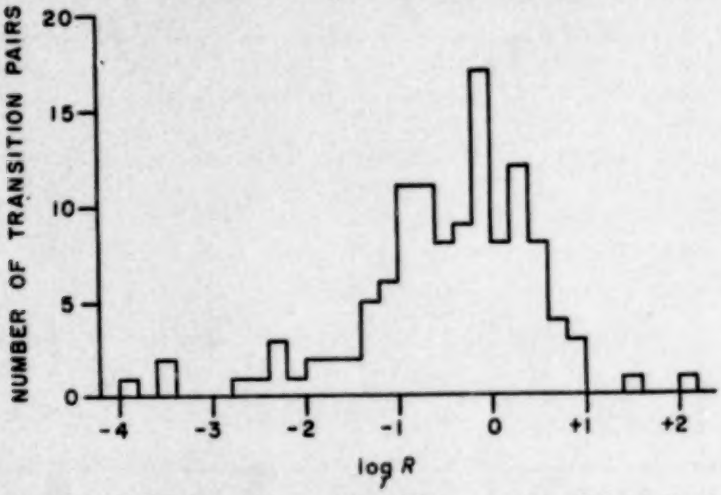

Fig. 1. Distribution in E1-E1 branching ratios. The quantity R is defined in the text.

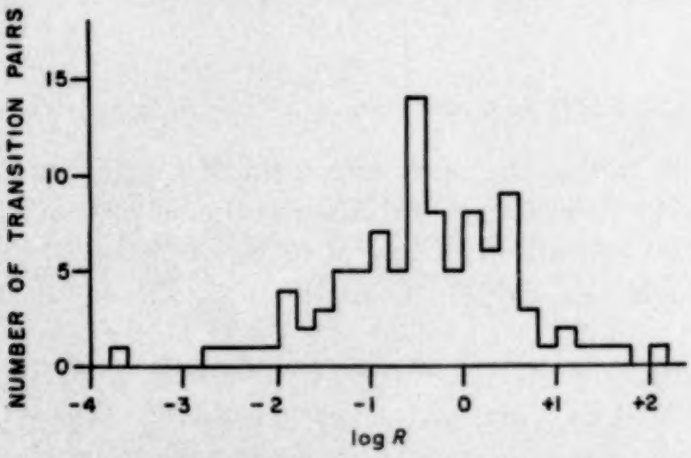

Fig. 2. Distribution in M1-M1 branching ratios.

† In cases in which two or more multipolarities are allowed for the same transition, the single particle model ordinarily predicts that the transition will proceed predominantly by way of the lower multipole radiation; and we have classified these transitions accordingly. Thus, for example, in the rather large number of cases in which a mixture of M1 and E2 radiation is possible, the transition is labelled as "predicted M1" even though it may contain an appreciable mixture of E2.

ratio of the higher energy gamma ray to the lower energy gamma ray has been taken in each case. The spread of the distribution is such that 76 % of the cases lie within a factor of 10 of $R = 1$ and 92 % within a factor of 100.

Fig. 2 shows the distribution in log R when both transitions are predicted †

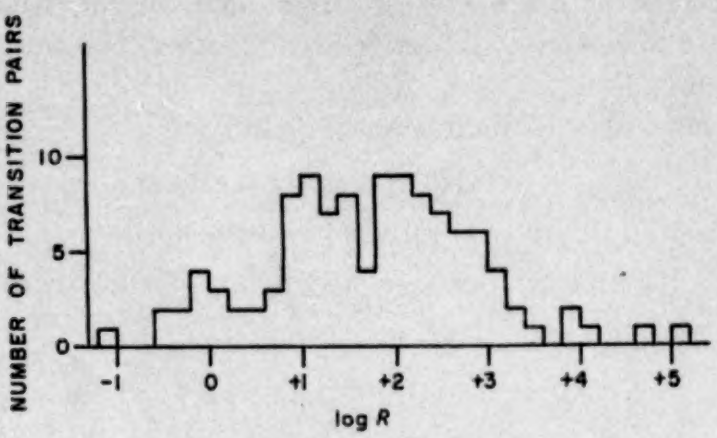

Fig. 3. Distribution in E2-M1 branching ratios.

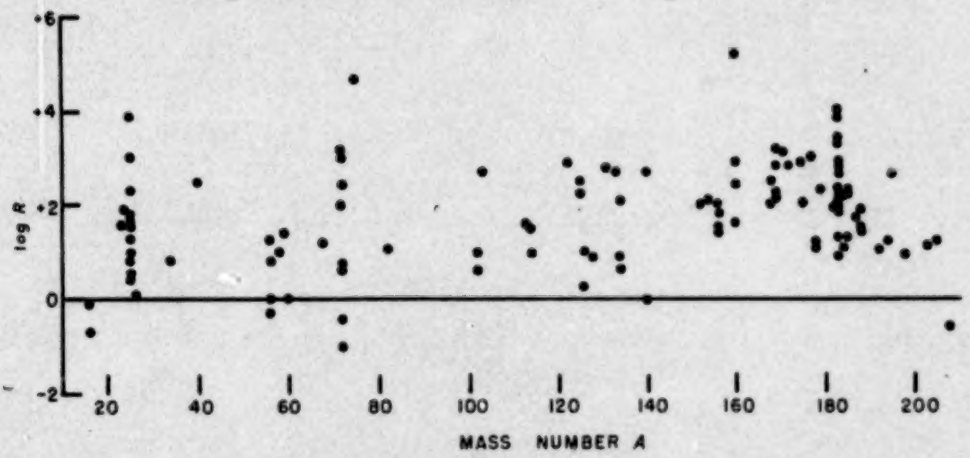

Fig. 4. E2-M1 branching ratios as a function of mass number.

to be M1, again taking the branching ratio of the higher energy to lower energy gamma ray in each case. This distribution is very similar to the other, with 69 % of the cases within a factor of 10 of $R = 1$ and 94 % within a factor of 100. Both the E1–E1 and M1–M1 distributions, although centred close to the value $R = 1$, show a distinct tendency for the higher energy gamma ray to be less intense relative to the lower energy gamma ray than is predicted by the single particle model: 69 % of the E1–E1 cases and 65 % of the M1–M1 cases have $R < 1$. This appears to indicate a systematic tendency for the E^3 dependence characteristic of dipole radiation in the single particle model to be over-ridden by other effects ††.

Fig. 3 shows the distribution in log R for branching ratios of E2 transitions

†† A least squares fit of the experimental branching ratios to the curve $I_1/I_2 = (E_1/E_2)^n$ gives $n = 0.11$ for E1 and $n = 1.1$ for M1 radiation.

to M1 transitions. The median is at log $R = 1.8$ ($R = 63$), showing very clearly the well-known enhancement of E2 transitions which is presumably associated with collective effects. This enhancement is effective throughout the entire range of atomic masses, as shown in fig. 4, in which log R is plotted as a function of mass number. The spread of the E2–M1 distribution is not very different from that of the previous two, 67 % of the cases lying within a factor of 10 of the median and 97 % within a factor of 100.

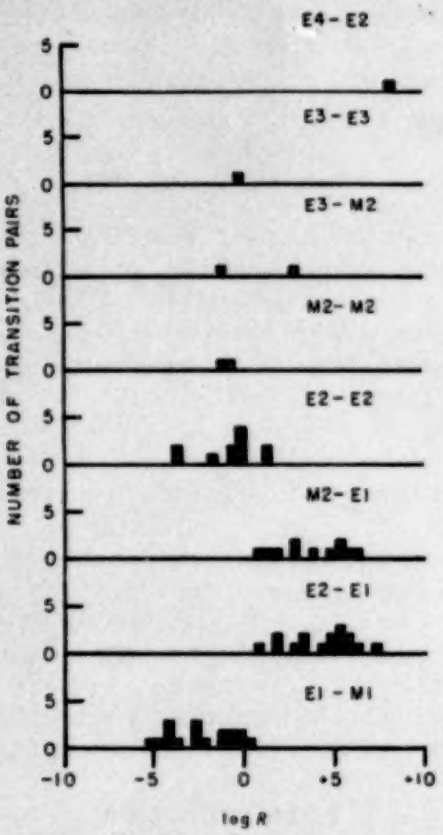

Fig. 5. Distributions in other branching ratios.

Data involving other types of transitions, shown in fig. 5, are not prolific enough to enable any very definite conclusions to be drawn concerning the median values or spreads of the distributions. One feature which is perhaps worthy of note is the weakness of the E1 transitions relative to their competitors as evidenced by the last three distributions in fig. 5.

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ANOMALOUS ELECTRIC-DIPOLE INTERNAL CONVERSION

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Abstract: A group of highly forbidden electric-dipole transitions in the heavy-element region are known to exhibit anomalous internal conversion. These transitions are close to or below the threshold for K conversion. Their L and M subshell conversions show anomalies which exhibit a definite pattern: L_I and L_{II} subshell conversion is enhanced relative to the corresponding "book" values, while L_{III} is approximately normal. This paper attempts to explain this pattern in terms of "penetration" nuclear matrix elements for internal conversion, which arise because of the penetration of the converting atomic electron within the nucleus. It is found that for E1 transitions there are two types of penetration matrix elements. One of these, $\int \mathbf{j}_n \cdot \hat{\mathbf{r}} r^2 Y_1^*$, gives an important contribution only to $s_{\frac{1}{2}} \leftrightarrow p_{\frac{1}{2}}$ electron transitions. Such electron transitions are present in L_I and L_{II} conversion, but are absent in L_{III} conversion. Since calculations based on approximate nuclear models indicate that this matrix element is much larger than other penetration matrix elements, a ready qualitative explanation of the experimental results for the three L subshells is available. Similar results are expected for the M subshells, etc. Quantitative interpretation of the observed conversion anomalies evaluates the nuclear penetration matrix elements. These matrix elements contain information in addition to that contained in the gamma-ray matrix element.

1. Introduction

Experimental data are available on the anomalous internal conversion of a group of low-energy electric-dipole transitions in heavy elements. Nilsson and Rasmussen [2]) correlated the observed anomalies with the degree of forbiddenness of the gamma-ray transitions and possible model-dependent selection rules for the various nuclear matrix elements involved. Asaro, Stephens, Hollander and Perlman [3]) have recently presented detailed experimental evidence on the conversion anomalies observed, and emphasize the following experimental facts: 1) the nuclear transitions are close to or below the thresholds for K conversion and show anomalous internal conversion in the L and possibly the M and higher shells; 2) the magnitudes of the anomalies increase with the degree

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††† Preliminary reports of this work are given in ref. [1]).

of forbiddenness of the gamma-ray transition; 3) in general, L_I and L_{II} conversion coefficients are enhanced relative to their "book" values, while L_{III} conversion is approximately its tabulated value; 4) this general pattern is found in both odd-proton and odd-neutron nuclei; 5) similar results are found for conversion in the M subshells and possibly higher atomic shells.

For example, in the case of the 84-keV transition in Pa^{231}, the observed L_I and L_{II} conversion coefficients are $\approx$ 20 times the corresponding "book" values for a pure E1 transition, while the L_{III} conversion coefficient equals its "book" value to within experimental error.

This paper presents a possible explanation of these observations in terms of the existing formalism for the internal conversion process, and provides a mechanism for the extraction and interpretation of the nuclear penetration matrix elements appearing in this formalism.

2. The Internal-Conversion Process

The two important modes of nuclear de-excitation, gamma-ray emission and internal conversion, are intimately related. In the limit of a point nucleus, the rates of these processes are proportional, and their ratio, the internal conversion coefficient

$$\alpha \equiv \mathcal{W}_e / \mathcal{W}_\gamma, \tag{1}$$

is independent of any nuclear matrix elements. For a finite nucleus, the converting atomic electron penetrates the nuclear region, and the rate of internal conversion then involves penetration nuclear matrix elements which are different from that determining the rate of gamma-ray emission [4]). In general, the effect of these penetration terms is small, and one may speak of an approximately nuclear-structure independent internal-conversion coefficient, such as have been extensively tabulated [5, 6]).

There are cases when the gamma-ray matrix element is appreciably smaller than the dimensional value, but, as we shall see, the penetration matrix elements are not. For these cases there may be large differences between empirical conversion coefficients or conversion ratios, and their tabulated or "book" values. One then speaks of "anomalous" internal conversion. In extreme cases the penetration effects may dominate.

Since penetration effects are determined by nuclear matrix elements which are different from the corresponding gamma-ray matrix element, a study of conversion anomalies leads to further information which is not available from the gamma-ray matrix elements themselves [4, 7, 9]).

3. Internal Conversion of E*L* Transitions

The rate of ejection of internal-conversion electrons is proportional to $\sum |U_{fi}|^2$, where U_{fi} is the matrix-element of the usual retarded electromagnetic inter-

action between the nuclear transition charges and currents, and those of the converting atomic electron. For EL transitions the separation of U_{fi} into gamma-ray and penetration terms was first performed by Nilsson and Rasmussen [2]) and Kramer [8]). This was carried through more completely and for all transitions by Green and Rose [9]). The result for EL conversion can be written in the especially useful form [1])†

$$U_{fi}(\mathrm{E}L) = B(\mathrm{E}L, M; e_i, e_f)\left[R_{\kappa_f}(e)\int_0^\infty \mathbf{j}_n \cdot \mathbf{A}^*_{LM}(k\mathbf{r}_n) + \int_0^\infty \mathbf{j}_n \cdot \nabla\Phi(r_n)Y^*_{LM}(\hat{\mathbf{r}}_n) + \int_0^\infty \mathbf{j}_n \cdot \frac{\hat{\mathbf{r}}_n}{r_n}\Theta(r_n)Y^*_{LM}(\hat{\mathbf{r}}_n)\right], \tag{2}$$

where the integration over the electron variables has been carried out, but the explicit dependence on the nuclear variable $\mathbf{r}_n$ and the nuclear transition current $\mathbf{j}_n$ remains. The common factor B is the result of integrating over the electron angular coordinates [1]) and the three functions $R_{\kappa_f}(e)$, $\Phi(r_n)$ and $\Theta(r_n)$ are derived from integrals over the electron radial variable. The significance of the three terms in eq. (2) is discussed below.

The first term in eq. (2) would be the only non-vanishing term if the converting electron never penetrated the nuclear region; for example, this would be the case in the limit of a point nucleus. This term exhibits the expected separation into electron and nuclear factors. It is important to note that the nuclear factor in this first term is identical with the usual EL gamma-ray matrix element

$$\int_0^\infty \mathbf{j}_n \cdot \mathbf{A}^*_{LM}(k\mathbf{r}_n) = \sqrt{\frac{2/\pi}{L(L+1)}}\int_0^\infty \mathbf{j}_n \cdot \left[\nabla \frac{\mathrm{d}}{\mathrm{d}r_n} r_n + k^2\mathbf{r}_n\right] j_L(kr_n)Y^*_{LM}(\hat{\mathbf{r}}_n), \tag{3}$$

where $j_L(kr_n)$ is the spherical Bessel function. In order to relate this matrix element to its textbook form, we consider the usual long-wave-length expansion in powers of $(kr)^2$:

$$\sqrt{\frac{2/\pi}{L(L+1)}}\,\frac{L+1}{(2L+1)!!}\,(kR)^L\left[\int_0^\infty \mathbf{j}_n \cdot \nabla\left(\frac{r}{R}\right)^L Y^*_{LM} - (kR)^2\left\{\frac{L+3}{2(L+1)(2L+3)}\int_0^\infty \mathbf{j}_n \cdot \nabla\left(\frac{r}{R}\right)^{L+2} Y^*_{LM} - \int_0^\infty \mathbf{j}_n \cdot \frac{\hat{\mathbf{r}}}{r}\left(\frac{r}{R}\right)^{L+2} Y^*_{LM}\right\} + \ldots\right]. \tag{4}$$

† Relativistic units, $\hbar = c = m_0 = 1$, are used throughout this paper. The quantity e is positive and $e^2 = \alpha \approx \frac{1}{137}$ in these units.

The first term is transformed into the conventional form by the use of the continuity equation

$$\nabla \cdot \mathbf{j}_n - ik\rho_n = 0, \tag{5}$$

which relates the nuclear transition current to the nuclear transition charge-density ρ_n. The resultant long-wave-length limit of the gamma-ray matrix element (the term of lowest order in $(kr)^2$) is then

$$\int_0^\infty \mathbf{j}_n \cdot \mathbf{A}^*_{LM}(k\mathbf{r}_n) \to -ik\sqrt{\frac{2/\pi}{L(L+1)}}\,\frac{L+1}{(2L+1)!!}\,(kR)^L \int_0^\infty \rho_n \left(\frac{r}{R}\right)^L Y^*_{LM}(\hat{\mathbf{r}}_n), \tag{6}$$

which is the form usually encountered.

The second and third terms in the expression for the conversion matrix element U_{fi}, eq. (2), arise entirely from the penetration of the converting atomic electron within the nuclear transition charge and current densities. There are two distinctly different types of penetration matrix elements for EL conversion: the $\mathbf{j} \cdot \nabla$ type and the $\mathbf{j} \cdot \hat{\mathbf{r}}$ type. Similar forms also appear in the gamma-ray matrix element (4), where, however, their small weighting is very different from that occurring in the conversion formula. In the case of conversion, the relative importance of the various terms involved depends on the initial and final state electron wave functions which enter into the functions R, Φ and Θ, explicit forms of which are given in appendix 1. These forms indicate that the two penetration terms in eq. (2) may be as much as a few percent of the first, the gamma-ray term, if all the nuclear matrix elements had their dimensional values. It may also be seen from these explicit forms that, barring cancellations (such as will be seen to occur for E1 transitions)

$$|\Theta/\Phi| \approx kR \approx \tfrac{1}{250}, \text{ dimensional value.} \tag{7}$$

Therefore, the intrinsic weighting of the $\mathbf{j} \cdot \nabla$ type of penetration term would generally be much greater than that of the $\mathbf{j} \cdot \hat{\mathbf{r}}$ type.

4. Internal Conversion of E1 Transitions

We confine our attention hereafter to the particular case of immediate experimental interest: E1 conversion in the three L shells. The consideration of other multipole orders and other atomic shells is straightforward, and has been given in part elsewhere [1, 9]†.

† The following general remarks can be made: E1 conversion anomalies are expected simultaneously in the K, L_I, M_I ... subshells and L_{II}, M_{II} ... subshells; the E1 conversion anomalies in the higher subshells, L_{III}, M_{III}, M_{IV} .... are expected to be much smaller. The conversion anomaiies for electric transitions higher than dipole need not conform to this dipole pattern.

The various initial and final electron states involved in E1 conversion in the three L shells are given in table 1. Since states of the lowest total angular momentum are most strongly represented at the nucleus, $s_{\frac{1}{2}} \leftrightarrow p_{\frac{1}{2}}$ electron transitions would be expected to have the largest weighting factors Φ and Θ.

TABLE 1

Initial and final electron states involved in the conversion of E1 transitions in the I, II and III subshells (e.g. L_I, L_{II}, L_{III}).

Initial state	κ_i	Final state	κ_f
I: $s_{\frac{1}{2}}$	-1	$p_{\frac{1}{2}}$	$+1^*$
		$p_{\frac{3}{2}}$	-2^*
II: $p_{\frac{1}{2}}$	$+1$	$s_{\frac{1}{2}}$	-1^*
		$d_{\frac{3}{2}}$	$+2^*$
III: $p_{\frac{3}{2}}$	-2	$s_{\frac{1}{2}}$	-1^*
		$d_{\frac{3}{2}}$	$+2$
		$d_{\frac{5}{2}}$	-3

The approximate but descriptive non-relativistic notation is given along with the corresponding relativistic κ values. The asterisks denote those electron transitions which involve the greatest penetrating weighting functions. This latter point is discussed in detail in appendices 1 and 2.

However, as Nilsson and Rasmussen have emphasized [2]), it is for just these $j = \frac{1}{2} \rightarrow j = \frac{1}{2}$ electron transitions that there is a strong cancellation between the component parts of the weighting function Φ †. This cancellation reduces the contribution of this term by an additional factor of the order of R, the nuclear radius, which even in heavy nuclei is only $\approx \frac{1}{50}$ (relativistic units). As a consequence, Φ for these $j = \frac{1}{2} \rightarrow j = \frac{1}{2}$ electron transitions is reduced to approximately the same order of magnitude as Φ for $j = \frac{1}{2} \rightarrow j = \frac{3}{2}$ transitions. Since $j = \frac{1}{2} \rightarrow j = \frac{1}{2}$ electron transitions occur in L_I and L_{II} conversion, and $j = \frac{1}{2} \leftrightarrow j = \frac{3}{2}$ transitions occur in conversion from all three subshells, the $\mathbf{j} \cdot \nabla$ type of penetration matrix element is weighted comparably in conversion in all three L subshells. Then, if there were only the $\mathbf{j} \cdot \nabla$ type of penetration terms, it would be difficult to account for the observed pattern of conversion anomalies in terms of penetration effects, since anomalies are observed in only the L_I and L_{II} subshells, but *not* in L_{III}.

However, the weighting function Θ which appears in the $\mathbf{j} \cdot \hat{\mathbf{r}}$ type of penetration term in (2) does not suffer any cancellation, and is therefore largest for the $j = \frac{1}{2} \rightarrow j = \frac{1}{2}$ electron transitions. In these cases it is found that the

† Such cancellations occur for any $\kappa_i = -\kappa_f$ transition, e.g. also for the $p_{\frac{3}{2}} \rightarrow d_{\frac{3}{2}}$ electron transition in L_{III} conversion. However, these terms are negligible for the present discussion.

dimensional estimate (7) is not realized, and in fact

$$|\Theta/\Phi| \approx 5k \approx 1 \qquad \text{for} \qquad s_{\frac{1}{2}} \leftrightarrow p_{\frac{1}{2}} \text{ electron transitions.} \tag{8}$$

For all other important electron transitions in table 1 the function Θ is much smaller than Φ†. In their elementary theory, Nilsson and Rasmussen neglected all terms of order k, and so effectively set $\Theta = 0$. However, the much greater relative importance of Θ in the cases of $s_{\frac{1}{2}} \leftrightarrow p_{\frac{1}{2}}$ electron transition opens a very important possibility.

The $\mathbf{j} \cdot \hat{\mathbf{r}}$ penetration matrix element, which is weighted by Θ, gives an important contribution only to the $s_{\frac{1}{2}} \leftrightarrow p_{\frac{1}{2}}$ electron transitions. Such transitions only occur for E1 conversion in the I and II subshells. The $\mathbf{j} \cdot \nabla$ penetration matrix element, on the other hand, appears comparable in conversion in all three subshells, I, II *and* III. The normal appearance of L_{III} conversion together with the anomalies in the L_I and L_{II} subshells would be explained if the $\mathbf{j} \cdot \hat{\mathbf{r}}$ type of penetration matrix element were intrinsically much larger than the $\mathbf{j} \cdot \nabla$ type. How this might be expected to be the case is discussed in the following section.

5. Nuclear Matrix Elements

We come now to the major point of interest — the significance of the two distinctly different penetration nuclear matrix elements appearing in eq. (2).

The specific form of these matrix elements is determined by the form of the radial weighting functions Φ and Θ. For low-energy nuclear transitions it is found that, to a sufficient approximation, Φ and Θ are proportional to r^3 in the important electron transitions††. The two most important penetration matrix elements are then of the form

$$\int \mathbf{j}_n \cdot \nabla r^3 Y_1^*, \qquad \int \mathbf{j}_n \cdot \frac{\hat{\mathbf{r}}}{r} r^3 Y_1^*. \tag{9}$$

It is important to emphasize that the two penetration matrix elements (9) and the corresponding gamma-ray matrix element, eq. (6) with $L = 1$, are related only within the framework of a specific nuclear model. One of these may tend to vanish, while the others may have their dimensional values. To illustrate this behaviour, we consider a simple set of assumptions. First, we take the nuclear current in the form of the usual convection and spin current

$$\mathbf{j}_n = \frac{e}{2iM} [\phi_f^* (\nabla \phi_i) - (\nabla \phi_f^*) \phi_i] + \mu \nabla \times (\phi_f^* \boldsymbol{\sigma} \phi_i), \tag{10}$$

† There is a cancellation effect in the negligible penetration term in the $p_{\frac{1}{2}} \rightarrow d_{\frac{1}{2}}$ electron transition in L_{III} conversion.

†† These penetration matrix elements appear for the electron transitions indicated by an asterisk in table 1. The corresponding forms for other cases involve at least two higher powers of $r \approx R \approx \frac{1}{50}$, and are therefore relatively weakly weighted in the conversion process. See appendices 1 and 2 for details.

where the ϕ are the nuclear wave functions, M is the nucleon mass and μ the nucleon magnetic moment. Second, we consider the nuclear wave functions to be of the strong-coupling form, appropriate to the asymptotic limit of large nuclear deformations [10, 11]). In this limit the intrinsic nuclear wave functions are characterized by the principal oscillator quantum number N, the oscillator quantum number along the axis of symmetry n and the projections Λ and Σ of the internal orbital and spin moments along this axis, respectively. Nilsson and Rasmussen [2]) have discussed the selection rules for E1 gamma-ray matrix elements and the $\mathbf{j} \cdot \nabla$ type of penetration term for this model. These asymptotic selection rules are given in appendix 4.

It is easily seen that the gamma-ray matrix element may be forbidden while the penetration matrix elements are allowed. For example, for the K-allowed transition having $\Delta N = 1$, $\Delta n = 2$, $\Delta\Lambda = 1$ and $\Delta\Sigma = 0$, the E1 gamma-ray matrix element vanishes because $\Delta n = 2$. However, because of the additional factor of r^2 the first penetration matrix element of the pair (9) would be allowed †.

Similar statements can be made about the second type of penetration matrix element (9). According to this model, whenever the first penetration matrix element is allowed, so is the second. The converse is not true, however. For example, in a spin-flip transition, $\Delta\Sigma = \pm 1$, the gamma-ray and $\mathbf{j} \cdot \nabla$ type of penetration matrix elements are both forbidden, while the $\mathbf{j} \cdot \hat{\mathbf{r}}$ type may be allowed, via the spin-current contribution. This is illustrated in greater detail below. The complete asymptotic selection rules for the $\mathbf{j} \cdot \hat{\mathbf{r}}$ type of penetration matrix element are given in appendix 4.

One can also see that the $\mathbf{j} \cdot \mathbf{r}$ type of penetration term is expected to be several orders of magnitude larger than the $\mathbf{j} \cdot \nabla$ type, even when both are allowed. Using the continuity eq. (5), the $\mathbf{j} \cdot \nabla$ type of penetration matrix element takes the form

$$\int \mathbf{j}_n \cdot \nabla r^3 Y_1^* = -ik \int \rho_n r^3 Y_1^*, \tag{11}$$

where k is the nuclear transition energy. It is important to note that the appearance of this factor k is a direct consequence of the continuity equation. In most of the experimental cases of interest, we have $k \lesssim 100$ keV$/m_e c^2$. On the other hand, the second type of penetration matrix element is expected to be of the order

$$\int \mathbf{j}_n \cdot \frac{\hat{\mathbf{r}}}{r} r^3 Y_1^* \approx -i\Delta E \int \rho_n r^3 Y_1^*, \tag{12}$$

† As detailed in appendix 4, the essential point is that the argument of the E1 gamma-ray matrix element $r^1 Y_1^*$ involves at most z^1, and hence cannot connect $|\Delta n| = 2$ levels. The penetration matrix element, on the other hand, involves $r^3 Y_1^*$ which contains an extra factor r^2 (including z^2) and can connect these levels.

where ΔE is some energy spacing characteristic of the dimensions of the nuclear model. For example, for a harmonic oscillator potential, ΔE would be the oscilllator spacing, which is $\approx 10\ \mathrm{MeV}/m_e c^2$. One then expects

$$\left|\int \mathbf{j}_n \cdot \frac{\hat{\mathbf{r}}}{r} r^3 Y_1^*\right| \Big/ \left|\int \mathbf{j}_n \cdot \nabla r^3 Y_1^*\right| \approx \frac{\Delta E}{k} \approx 10^2. \tag{13}$$

This situation provides a qualitative explanation of the appearance of appreciable conversion anomalies in only the L_I and L_{II} subshells.

To verify the result (13) by an explicit calculation, we consider a very simple version of the Nilsson model, a version invoked for illustration only. Consider first a single proton in an axially deformed harmonic-oscillator potential, with no other forces. The degree of distortion is chosen so that the level, $N+1, n+2, \Lambda+1, \Sigma$, lies a small energy, k, above the level N, n, Λ, Σ. The current appropriate to this Hamiltonian is simply the convection and spin current (10), since this form of the current automatically obeys the continuity eq. (5)†. The gamma-ray matrix element and the spin-current contribution to both penetration terms vanish here because $\Delta n = 2$. However, the convection current contributions to the two penetration terms are finite, and their ratio is

$$\frac{\int \mathbf{j}_n \cdot \frac{\mathbf{r}}{r} r^3 Y_1^*}{\int \mathbf{j}_n \cdot \nabla r^3 Y_1^*} = \frac{1}{Mk} \frac{\left(n\left|z\frac{\partial}{\partial z}\right|n+2\right)}{(n|z^2|n+2)} \frac{(N-n|x-iy|N-n-1)}{(N-n|x-iy|N-n-1)}. \tag{14}$$

The numerator is simplified by the identity

$$z\frac{\partial}{\partial z} = -\tfrac{1}{2}M[H_z, z^2]-\tfrac{1}{2}, \tag{15}$$

where H_z is the one-dimensional harmonic-oscillator Hamiltonian with eigenvalues ω_z, describing the oscillator motion along the axis of deformation. The ratio of the two penetration matrix elements for the model is then simply ††

$$\int \mathbf{j}_n \cdot \frac{\hat{\mathbf{r}}}{r} r^3 Y_1^* \Big/ \int \mathbf{j}_n \cdot \nabla r^3 Y_1^* = \frac{\omega_z}{k}. \tag{16}$$

The result (16) verifies the previous assertion regarding the relative orders of magnitude of the two types of penetration matrix elements. Most importantly, it shows that the delicate cancellations in the $\mathbf{j}\cdot\nabla$ type of matrix elements, which are required by the continuity equation, need have no counter-

† In using a more detailed nuclear model to calculate the magnitude of the penetration matrix elements, it is important to use that form of the nuclear current which satisfies the continuity eq. (5). See the remarks at the end of appendix 4.

†† The explicit value of the indicated intrinsic matrix elements is proportional to $\sqrt{N-n-\Lambda}$, which may vanish for particular orbital assignments because of a "radial" selection rule.

part in the evaluation of the $\mathbf{j}\cdot\hat{\mathbf{r}}$ type of penetration matrix elements. Different aspects of the nuclear current enter into the evaluation of these types of matrix elements. A conspicious illustration of this fact is that the spin current can make no contribution to the $\mathbf{j}\cdot\nabla$ type of matrix elements since $\nabla\cdot\mathbf{j}_{\text{spin}}=0$, while it may contribute importantly in the $\mathbf{j}\cdot\hat{\mathbf{r}}$ type.

For odd-neutron transitions there are other reasons for expecting a small value for the $\mathbf{j}\cdot\nabla$ type of nuclear penetration matrix elements; the one-particle neutron transition charge density is zero. Recoil effects, which are so important in the gamma-ray matrix element, are small in the penetration term (9a) because of the additional factors of r^2. On the other hand, one expects a sizeable contribution to the $\mathbf{j}\cdot\hat{\mathbf{r}}$ type of nuclear matrix element because of the neutron magnetic moment. The importance of this fact in internal conversion has been noted by Voikhanskii and Listengarten [12]). Employing the simplified model already used to illustrate the odd-proton case, it will be seen that when allowed, the odd-neutron spin-current contribution may be of the same order as the odd-proton convection-current contribution to the $\mathbf{j}\cdot\hat{\mathbf{r}}$ penetration term.

To illustrate this, consider the transition of an odd neutron from the state N, n, Λ, Σ, to the state $N-1, n, \Lambda-1, \Sigma+1$. Here both the gamma-ray and $\mathbf{j}\cdot\nabla$ type of penetration terms vanish because a spin flip is involved. However, the spin current does give a finite contribution to the $\mathbf{j}\cdot\hat{\mathbf{r}}$ type of penetration term. In order to appreciate the magnitude of this contribution we may compare this result with that already obtained for the odd-proton convection-current result, even though these two situations involve different kinds of particles and transitions between levels which are necessarily different. The ratio is †

$$\left|\frac{\left(\int \mathbf{j}_n\cdot\frac{\hat{\mathbf{r}}}{r}r^3Y_1^*\right)_{\text{odd-neutron spin current}}}{\left(\int \mathbf{j}_n\cdot\frac{\hat{\mathbf{r}}}{r}r^3Y_1^*\right)_{\text{odd-proton convection current}}}\right|=\frac{\mu_n}{\sqrt{n+1}\,\sqrt{n+2}}. \tag{17}$$

For $\mu_n\approx 2$ nuclear Bohr magnetons and $n\approx 2$, this ratio is $\approx\frac{1}{2}$. A similar analysis indicates that in the odd-proton case the spin-current contribution to the type of penetration term may be of the same order as its convection-current contribution.

The above results illustrate the fact that in both odd-neutron and odd-proton transitions, spin flip or not, the penetration matrix elements may be allowed even though the gamma-ray matrix element is strongly inhibited. Of the two types of penetration matrix elements, the $\mathbf{j}\cdot\hat{\mathbf{r}}$ type is *always* expected

† Note again that the matrix elements are proportional to $\sqrt{N-n-\Lambda}$, and so may vanish for a particular set of quantum numbers.

to be much larger than the $\mathbf{j} \cdot \nabla$ type for these low-energy, particle transitions. This situation is just that which we have already noted is sufficient to explain the general pattern of the experimental conversion anomalies, and further, to explain them in a manner which is essentially independent of whether the gamma-ray transition is inhibited by radial or spin selection rules.

In the following section we analyze a specific experimental case, extract the two penetration matrix elements, and compare them with the above theoretical considerations.

6. Analysis of Experiment

In order to interpret conversion anomalies in detail, the contributions from the gamma-ray term and the penetration terms in (2) must be separated. To do this we must know the complex radial integrals $R_{\kappa_f}(e)$, which weight the gamma-ray contributions to the conversion, as well as the penetration weighting functions Φ and Θ. The evaluation of these two penetration functions is straightforward, and is discussed in detail in appendix 1. The evaluation of the radial electron integrals $R_{\kappa_f}(e)$, on the other hand, necessitates machine computation, and those required for the L shells are not available in the published literature. A detailed analysis of the experimentally observed L-shell anomalies is therefore impossible at the present time †.

However, in the limit where the contribution of the gamma-ray term in eq. (2) is much less than that of the penetration terms, the situation becomes amenable to analysis. In this "penetration-conversion limit", the gamma-ray term in eq. (2) is neglected relative to the penetration terms, and the rate of

TABLE 2

The experimental L subshell conversion properties of the 84.2-keV transition in Pa^{231} as taken from the data reported by Asaro, Stephens, Hollander and Perlman [3])

Subshell (L_i)	Measured ICC (α_{L_i})	Conversion anomaly
L_I	1.3 ±0.2	22 ±5
L_{II}	0.65 ±0.15	17 ±5
L_{III}	0.046±0.014	1.2±0.4

The derivation of the corresponding electron transition rates W_e from these and additional data is discussed in the footnote on p. 613. The gamma-ray halflife is $0.693/W_\gamma = 5.68 \times 10^{-7}$ sec.

† *Note added in proof:* The radial integrals have been made available to us by Professor L. A. Sliv. A preliminary analysis of the Pa^{231} data, using these integrals, does not qualitatively differ from the approximate analysis given below. A full analysis of this and the other cases wil be presented elsewhere.

conversion-electron ejection depends only on the nuclear penetration terms †. Of the presently available experimental cases, only the L_I and L_{II} conversion of the 84.2-keV transition in Pa^{231} approach this limit, and so it is only these transitions which we can consider at the moment.

The experimental properties of the 84.2-keV transition in Pa^{231} are presented in table 2. The L_I and L_{II} conversion coefficients are seen to be ≈ 20 times their book values, while L_{III} is that expected for a pure E1 transition, within experimental error. This normal L_{III} conversion coefficient immediately excludes an explanation of the L_I and L_{II} conversion anomalies in terms of the admixture of higher multipoles, and we take this transition to be pure E1 for the approximate analysis considered below. Since the L_I and L_{II} conversion anomalies are ≈ 20, the penetration contributions are $\approx \sqrt{20}$ times as large as the gamma-ray contribution to their conversion matrix elements. These two electron transitions do then roughly approximate the complete penetration-conversion limit.

We can now write the rates of L_I and L_{II} conversion-electron ejection in the penetration-conversion limit. For the particular case of the 84.2-keV E1 transition in Pa^{231} these are, as shown in appendix 3,

$$\mathscr{W}_e(L_I) \approx 3.48\times 10^{-3}\left[7.97(\|\mathbf{j}\cdot\nabla r^3 Y_1^*\|) + 16.79\left(\left\|\mathbf{j}\cdot\frac{\hat{\mathbf{r}}}{r}r^3 Y_1^*\right\|\right)\right]\frac{m_e c^2}{\hbar}\ \text{sec}^{-1}, \tag{18}$$

$$\mathscr{W}_e(L_{II}) \approx 3.48\times 10^{-3}\left[14.20(\|\mathbf{j}\cdot\nabla r^3 Y_1^*\|) - 12.45\left(\left\|\mathbf{j}\cdot\frac{\hat{\mathbf{r}}}{r}r^3 Y_1^*\right\|\right)\right]\frac{m_e c^2}{\hbar}\ \text{sec}^{-1}. \tag{19}$$

The leading terms of the two types of penetration matrix elements (9) appear in (18) and (19) in the form of the usual reduced matrix elements ††. As discussed in the appendices, there are, in fact, additional moderately-weighted matrix elements involving higher powers of r^2 which have been omitted in the present discussion for simplicity.

The experimental values of the rates of L_I and L_{II} conversion-electron ejection as derived from the experimental data given by Asaro *et al.*[3]) are

† Because the radial weighting factors R, Φ and Θ are different for each electron transition, the approach to the penetration-conversion limit must be considered separately for each component electron transition in the conversion. For example, in the case of the 84.2-keV transition in Pa^{231}, discussed in the text, only the $s_{\frac{1}{2}} \leftrightarrow p_{\frac{1}{2}}$ electron transitions apparently approximate this limit while the remaining five electron transitions listed in table 1 approximate their book behaviour.

†† The reduced matrix element is the usual matrix element, e.g., in the form of (9), with the factor containing the magnetic quantum numbers, a Clebsch-Gordan coefficient, explicitly divided out. See eq. (39).

approximately †

$$\mathscr{W}_e(L_I) \approx (1.6\pm0.8)\times10^6\,\mathrm{sec}^{-1}, \qquad \mathscr{W}_e(L_{II}) \approx (0.8\pm0.4)\times10^6\,\mathrm{sec}^{-1}. \quad (20)$$

Combining these experimental data with the expressions (18) and (19), we find two sets of penetration matrix elements consistent with the experimental data:

$$(\,\|\mathbf{j}\cdot\nabla r^3Y_1^*\|\,) = \pm5.5\times10^{-8}, \qquad \left(\left\|\mathbf{j}\cdot\frac{\hat{\mathbf{r}}}{r}r^3Y_1^*\right\|\right) = \pm2.0\times10^{-8}, \quad (21)$$

and

$$(\,\|\mathbf{j}\cdot\nabla r^3Y_1^*\|\,) = 0, \qquad \left(\left\|\mathbf{j}\cdot\frac{\hat{\mathbf{r}}}{r}r^3Y_1^*\right\|\right) = \pm4.5\times10^{-8}. \quad (22)$$

The uncertainties in these matrix elements are $\approx\pm1\times10^{-8}$. Of the two possibilities (21) and (22), the first is inconsistent with the observed normal L_{III} conversion coefficient and is therefore dropped. The second solution, (22), gives normal L_{III} conversion. It is to be noted that it satisfies the relation

$$\left|\frac{\int\mathbf{j}\cdot\nabla r^3Y_1^*}{\int\mathbf{j}\cdot\frac{\hat{\mathbf{r}}}{r}r^3Y_1^*}\right| = \left|\frac{(\,\|\mathbf{j}\cdot\nabla r^3Y_1^*\|\,)}{\left(\left\|\mathbf{j}\cdot\frac{\hat{\mathbf{r}}}{r}r^3Y_1^*\right\|\right)}\right| \ll 1, \quad (23)$$

anticipated on theoretical grounds.

The absolute magnitudes of the two penetration matrix elements (22) are now also available for comparison with the predictions of a nuclear model. The first important fact to note is that the values as deduced above are small compared with their dimensional values, kR^3 for the $\mathbf{j}\cdot\nabla$ type of penetration matrix element, and the much larger value ωR^3 for the $\mathbf{j}\cdot\hat{\mathbf{r}}$ type:

$$\left|\frac{(\,\|\mathbf{j}\cdot\nabla r^3Y_1^*\|\,)}{kR^3}\right| \leqq \tfrac{1}{10}, \qquad \left|\frac{\left(\left\|\mathbf{j}\cdot\frac{\hat{\mathbf{r}}}{r}r^3Y_1^*\right\|\right)}{\omega R^3}\right| \approx \tfrac{1}{200}, \quad (24)$$

where we have taken $\omega = 80/A^{\frac{1}{3}}$. In comparison, the observed gamma-ray matrix element is only $\approx 2\times10^{-4}$ of its dimensional value of kR. For further understanding we must turn to detailed nuclear models.

The Mottelson-Nilsson [3, 13]) assignments of the levels involved in the 84.2-keV transition in Pa^{231} involve the mixture of states of different K values. Transitions between some of these levels are K forbidden, while others are K allowed. The K selection rules apply to both the gamma-ray and penetration matrix elements. Since the K mixing is only qualitatively understood, it is

† The central and extreme values of the electron transition probabilities deduced from the experimental data given in ref. [3]) are $\mathscr{W}_e(L_I) = (1.59, +2.96, -0.84)\times10^6\,\mathrm{sec}^{-1}$ and $\mathscr{W}_e(L_{II}) = (0.79, +1.58, -0.39)\times10^6\,\mathrm{sec}^{-1}$. For the present purposes we have adopted the approximate values (20).

fruitless to attempt any detailed calculations of the penetration matrix elements involved. However, it is of interest to note that the intrinsic assignments of the K-allowed components forbid the gamma-ray matrix element, but allow the penetration matrix elements. The proposed admixture of levels is therefore in agreement with the fact that the observed penetration matrix elements are less hindered than the corresponding gamma-ray matrix element.

It is unfortunate that the case of the 84.2-keV transition in Pa^{231} just discussed — the only case which can be even approximately analyzed at the moment — is not interpreted in terms of a simple model. However, as soon as the radial electron matrix elements for L-shell conversion are available, a complete analysis of the experimental data for the other experimentally observed cases of anomalous L conversion will be undertaken, and a more detailed comparison with model predictions will then be made.

7. Directional-Correlation Experiments

The presence of penetration matrix elements also affects the directional correlation of the emitted conversion electrons. This occurs through the change of the relative amplitudes of the various final electron states (table 1) which depend, in turn, on exactly the same expressions for the conversion matrix element (2), which we have already discussed. As for the total conversion coefficient, the prediction of the effect of the penetration terms on the various correlation parameters depends on a knowledge of the radial electron integrals $R_{\kappa_f}(e)$, which determine the gamma-ray contributions to the conversion process. Since these are presently unavailable to us, no detailed predictions of the dependence of the electron particle parameters on the penetration matrix elements can now be made.

However, in the penetration-conversion-limit defined in the previous section†, relatively simple predictions are again possible. In general, the expected dominance of the $\mathbf{j}\cdot\hat{\mathbf{r}}$ penetration term involves predominant ejection of electrons into $j=\frac{1}{2}$ states: $p_{\frac{1}{2}}$ from the L_I shell and $s_{\frac{1}{2}}$ from the L_{II} shell. Since $j=\frac{1}{2}$ electrons lead to isotropy, the observed conversion-electron anisotropies for the case of the 84.2-keV transition in Pa^{231}, for example, should be less than their book values for L_I and L_{II} electrons, but approximately equal to their book value for L_{III} electrons.

The measurement of such electron correlations complements the measurement of the conversion coefficients themselves, and provides further information for determining the values of the penetration matrix elements involved.

We thank Professor Niels Bohr and his staff for the generous hospitality and facilities of the Institute for Theoretical Physics, where the present work was completed. We also express our gratitude for the many other expressions of Danish hospitality which have served to make our stay in Denmark both possible and so pleasant.

† See the second footnote in sect. 6.

Appendix 1

THE FUNCTIONS R, Φ AND Θ

If the initial and final state Dirac electron wave functions are taken in the usual form

$$\psi_e = \begin{pmatrix} -i\chi_{-\kappa} f_\kappa \\ \chi_\kappa g_\kappa \end{pmatrix}, \tag{25}$$

the radial electron integral characterizing the gamma-ray contributions to the conversion matrix element for the final electron state κ_f is

$$R_{\kappa_f} = (\kappa_1 - \kappa_f) \int_0^\infty r'^2 \mathrm{d}r' (g_f f_1 + f_f g_1) h^{(1)}_{L-1}(kr') + L \int_0^\infty r'^2 \mathrm{d}r' [(g_f f_1 - f_f g_1) h^{(1)}_{L-1}(kr') + (f_f f_1 + g_f g_1) h^{(1)}_L(kr')]. \tag{26}$$

The functions (26) have been published by Rose for the particular case of K conversion for an unscreened point nucleus [6]). The separate functions appropriate for the L shell are not now available in the literature.

The penetration weighting functions Φ and Θ are [1])

$$\Phi(r) = \frac{1/k}{\sqrt{L(L+1)}} \left[\left(\frac{\mathrm{d}}{\mathrm{d}r} r h^{(1)}_L(kr) \int_0^r r'^2 \mathrm{d}r' \mathcal{O} r' j_L(kr') - \left(\frac{\mathrm{d}}{\mathrm{d}r} r j_L(kr) \right) \int_0^r r'^2 \mathrm{d}r' \mathcal{O} r' h^{(1)}_L(kr') \right], \tag{27}$$

$$\Theta(r) = \frac{1/k}{\sqrt{L(L+1)}} \left[-i(g_f f_1 - f_f g_1) r^3 + (kr)^2 h^{(1)}_L(kr) \int_0^r r'^2 \mathrm{d}r' \mathcal{O} r' j_L(kr') - (kr)^2 j_L(kr) \int_0^r r'^2 \mathrm{d}r' \mathcal{O} r' h^{(1)}_L(kr') \right], \tag{28}$$

where

$$\mathcal{O} = (g_f f_1 - f_f g_1) k - (f_f f_1 + g_f g_1) \frac{\mathrm{d}}{\mathrm{d}r}. \tag{29}$$

To simplify the above we make use of the long-wave-length expansions for the spherical Bessel and Hankel functions

$$j_L(kr) \approx \frac{(kr)^L}{(2L+1)!!} + \ldots, \quad h^{(1)}_L(kr) \approx -i \frac{(2L-1)!!}{(kr)^{L+1}} + \ldots, \tag{30}$$

which give

$$\Phi(r) \approx \frac{i}{k} \frac{\sqrt{L(L+1)}}{2L+1} \frac{r^3}{kr} \left\{ r^{L-2} \int_0^r \frac{\mathrm{d}r'}{r'^{L-1}} (f_f f_1 + g_f g_1) - \frac{1}{r^{L+3}} \int_0^r r'^{L+2} \mathrm{d}r' (f_f f_1 + g_f g_1) + kr \left[\frac{1}{L} r^{L-3} \int_0^r \frac{\mathrm{d}r'}{r'^{L-2}} (g_f f_1 - f_f g_1) + \frac{1}{L+1} \frac{1}{r^{L+4}} \int_0^r r'^{L+3} \mathrm{d}r' (g_f f_1 - f_f g_1) \right] + \mathrm{O}(kr)^2 \right\}, \tag{31}$$

$$\Theta(r) \approx + \frac{i}{k} \frac{\sqrt{L(L+1)}}{2L+1} \frac{r^3}{1} \left\{ - \frac{2L+1}{L(L+1)} (g_t f_1 - f_t g_1) + O(kr) \right\}. \quad (32)$$

The estimate (7) follows from (31) and (32) if all the indicated radial integrals had their order of magnitude values.

The next step in the evaluation of the functions Φ and Θ involves the substitution of explicit forms for the radial electron wave functions f_κ and g_κ within the nucleus. For this purpose we make use of the expansions discussed by Matumoto and Yamada [14]), the leading terms of which are given below. For the "large "components one has

$$\begin{aligned} g_{-|\kappa|}(r) &= g_{-|\kappa|}(0) r^{|\kappa|-1} \left[J_{|\kappa|}(r) - \frac{W+1}{2|\kappa|-1} r^2 \phi_{|\kappa|}(r) + \ldots \right], \\ f_{+|\kappa|}(r) &= f_{+|\kappa|}(0) r^{|\kappa|-1} \left[J_{|\kappa|}(r) - \frac{W-1}{2|\kappa|-1} r^2 \phi_{|\kappa|}(r) + \ldots \right], \end{aligned} \quad (33)$$

and for the "small" components

$$\begin{aligned} f_{-|\kappa|}(r) &= g_{-|\kappa|}(0) r^{|\kappa|} \left[\phi_{|\kappa|}(r) - \frac{W-1}{2|\kappa|+1} J_{|\kappa|}(r) + \ldots \right], \\ g_{+|\kappa|}(r) &= -f_{+|\kappa|}(0) r^{|\kappa|} \left[\phi_{|\kappa|}(r) - \frac{W+1}{2|\kappa|+1} J_{|\kappa|}(r) + \ldots \right], \end{aligned} \quad (34)$$

where the quantities $f_\kappa(0)$ and $g_\kappa(0)$ are constants, W is the total energy of the electrons (bound or free) and κ the usual relativistic quantum number.

The explicit forms of $J_{|\kappa|}$ and $\phi_{|\kappa|}$ depend on the specific assumptions made regarding the static nuclear charge distribution. In general, however, their orders of magnitude are

$$\phi_{|\kappa|} \approx - \frac{\alpha Z}{2R}, \qquad J_{|\kappa|} \approx 1, \quad (35)$$

where R is some nuclear radius (in relativistic units). Thus, the first terms in (33) and (34) dominate the second ones by a factor of at least 10—20 in heavy elements, and it is therefore usually sufficient to consider only these first terms in a semi-quantitative discussion. However, for transitions for which $\kappa_f = -\kappa_1$ there is a cancellation among these leading terms in the electron wave-function combination $(f_t f_1 + g_t g_1)$, and the higher terms in (33) and (34) must necessarily be included. Specific results are given below for a uniformly charged nucleus. The relevant functions $J_{|\kappa|}(r)$ and $\phi_{|\kappa|}(r)$ are then

$$\begin{aligned} J_{|\kappa|}(r) \approx \Bigg\{ 1 - (\alpha Z)^2 \Bigg[& \frac{\frac{9}{16}}{2|\kappa|+1} \frac{r^2}{R^2} - \frac{\frac{3}{16}}{} \left(\frac{1}{2|\kappa|+1} + \frac{1}{2|\kappa|+3} \right) \frac{r^4}{R^4} \\ & + \frac{\frac{1}{24}}{2|\kappa|+3} \frac{r^6}{R^6} \Bigg] + O(\alpha Z)^4 \Bigg\}, \end{aligned} \quad (36)$$

$$\phi_{|\kappa|}(r) \approx -\frac{\alpha Z}{2R}\left\{\left[\frac{3}{2|\kappa|+1}-\frac{1}{2|\kappa|+3}\frac{r^2}{R^2}\right]+O(\alpha Z)^2\right\}. \tag{37}$$

Substituting (36) and (37) in (31) and (32) the penetration weighting functions Φ and Θ take the form of power series in r^2. For simplicity, only the terms involving the lowest power of r, are considered below, except in the case of Θ in the critical $j=\frac{1}{2}\to j=\frac{1}{2}$ electron transitions, where the lowest two powers are kept. These leading terms for the penetration weighting functions are given in table 3 for the special case of E1 transitions, i.e. $L=1$.

TABLE 3

Expressions for the leading terms in the expansion of the penetration weighting functions Φ and Φ appropriate for E1 conversion in the I, II and III subshells (e.g. L_I, L_{II}, L_{III}).

κ_i	κ_f	$\Phi(r)\Big/\frac{i}{k^3}\frac{\sqrt{2}}{3}$	$\Theta(r)\Big/\frac{i}{k\sqrt{2}}$
-1	$+1$	$+g_{-1}(0)f_{+1}(0)[2-5k]\frac{1}{10}r^3$	$+g_{-1}(0)f_{+1}(0)r^3\left[1-\frac{(\alpha Z)^2}{2}\frac{r^2}{R^2}\right]$
	-2	$+g_{-1}(0)g_{-2}(0)\frac{3}{10}r^3$	$+g_{-1}(0)g_{-2}(0)\frac{\alpha Z}{5R}r^5$
$+1$	-1	$+f_{+1}(0)g_{-1}(0)[2+5k]\frac{1}{10}r^3$	$-f_{+1}(0)g_{-1}(0)r^3\left[1-\frac{(\alpha Z)^2}{2}\frac{r^2}{R^2}\right]$
	$+2$	$+f_{+1}(0)f_{+2}(0)\frac{3}{10}r^3$	$+f_{+1}(0)f_{+2}(0)\frac{\alpha Z}{5R}r^5$
-2	-1	$+g_{-2}(0)g_{-1}(0)\frac{3}{10}r^3$	$-g_{-2}(0)g_{-1}(0)\frac{\alpha Z}{5R}r^5$
	$+2$	$+g_{-2}(0)f_{+2}(0)\frac{3}{70}[1-7k]r^5$	$+g_{-2}(0)f_{+2}(0)r^5$
	-3	$+g_{-2}(0)g_{-3}(0)\frac{3}{28}r^5$	$+g_{-2}(0)g_{-3}(0)\frac{\alpha Z}{5R}\frac{3}{7}r^7$

The format of this table parallels that of table 1, and the notation is defined in appendix 1. For the critical $s_{\frac{1}{2}}\leftrightarrow p_{\frac{1}{2}}$ electron transitions, the two leading terms of Θ are given.

The results in table 3 indicate that the approximate relation (7) is satisfied in all cases except the three critical transitions for which $\kappa_f=-\kappa_i$. In these latter cases the ratio of Θ to Φ is much larger than $kR\approx\frac{1}{250}$. In particular, for $k=0.2$ this ratio is ≈ 1 for the $s_{\frac{1}{2}}\leftrightarrow p_{\frac{1}{2}}$ electron transitions.

It must be emphasized that the terms listed in table 3 represent only the lowest terms in the power expansion of the penetration weighting functions. There may be higher-order terms which are moderately weighted in the expansion. For example, the r^5 terms in Θ for the $j=\frac{1}{2}\to j=\frac{1}{2}$ electron transitions which are given explicitly in table 3 are weighted by about 20 % of the

r^3 terms, in heavy elements. These higher terms involve higher powers of the nuclear radial variable r and, hence, different penetration nuclear matrix elements with different selection rules, such as are discussed in appendix 4.

The effect of these higher terms is not considered in detail here only because it has not been necessary to invoke them in the present preliminary analysis, and not because it is assumed that their cumulative quantitative contribution need always be small.

Appendix 2

EVALUATION AND APPLICATION TO Pa^{231}

In order to evaluate the penetration weighting functions listed in table 3 we must know the constants $f_{|\kappa|}(0)$ and $g_{|\kappa|}(0)$. These are most easily evaluated by first expressing them in terms of the corresponding values of the electron wave functions at the nuclear surface, using the expressions (33). In principle, the values of the electron wave functions at the nuclear surface are then determined by matching the internal solutions with the external ones, and appropriately normalizing the whole.

For the present purposes, the numerical values of the electron wave functions have been obtained in the following ways. The bound-state wave functions for the three L subshells have been interpolated from the results of Brysk and Rose [15]), and the continuum-state wave functions have been computed analytically [16]) for a uniformly charged nucleus † of radius $R = 1.20\ A^{\frac{1}{3}}$ fm. The specific numerical results used in the following discussions are presented in table 4. The results of Brysk and Rose include the effects of the finite nuclear size and atomic screening on the bound state wave functions.

TABLE 4

Values of the constants $f(0)$ and $g(0)$ characterizing the weighting of the "large" components of the initial (bound-state), and final (continuum-state), electron wave functions at the centre of the nucleus, appropriate for the conversion of the 84.2 keV transition in Pa^{231}.

Factor	Bound state	Continuum state
$g_{-1}(0)$	+1.356	+6.33
$f_{+1}(0)$	−0.457	−2.89
$g_{-2}(0)$	+0.132	+1.16
$f_{+2}(0)$		−0.344
$g_{-3}(0)$		+0.118

These quantities are defined by (33) in appendix 1, and their evaluation is discussed in appendix 2. The indicated results for the continuum functions were calculated for an average L-shell binding energy of 19.3 keV.

† Rose [16]) gives the regular solutions for a point nucleus. The corresponding values for a finite nucleus were obtained by matching a linear combination of regular and irregular solutions with the internal solutions (33) and (34) at the nuclear surface (see ref. [1]) for details).

Our analytic results for the continuum-state wave functions include the effects of the finite nuclear size in an approximate way, but neglect possible effects of atomic screening. It is expected that the overall accuracy of these continuum functions lies within 5—10 %, which is sufficient for the present purposes.

TABLE 5

Values of the leading terms in the expansion of the penetrating weighting functions Φ and Θ appropriate for the conversion of the 84.2-keV E1 transition in the three L subshells of Pa^{231}.

κ_1	κ_t	$\Phi(r)$	$\Theta(r)$
-1	$+1$	$-7.97\, ir^3$	$-16.79\, ir^3[1-0.22r^2/R^2]$
	-2	$+\ 8.14\, ir^3$	$+46.8\ \ ir^5$
$+1$	-1	$-14.2\, ir^3$	$+12.45ir^3[1-0.22r^2/R^2]$
	$+2$	$+\ 0.817\, ir^3$	$+\ 4.68\, ir^5$
-2	-1	$+\ 4.37\, ir^3$	$-25.1\ \ ir^5$
	$+2$	$+\ 5.23\, i\times 10^{-3} r^5$	$-\ 0.195\, ir^5$
	-3	$+\ 2.92\, i\times 10^{-2} r^5$	$+\ 0.201\, ir^7$

The format of this table is identical with that of table 3.

Substituting the values in table 4 into the expressions in table 3, we obtain the results given in table 5 for the particular case of interest; the 84.2-keV E1 transition in Pa^{231}.

Appendix 3

CONVERSION TRANSITION RATES

The general expression for the rate of ejection of EL conversion electrons is [6, 9])

$$\mathscr{W}_e = \frac{2\pi\alpha}{2I_1+1} \sum_{M_1, M_t} \sum_{m_1, m_t} \sum_{\kappa_t} |U_{fi}|^2, \tag{38}$$

where M and m denote the nuclear and electron quantum numbers and I the nuclear spin. The sums over the magnetic quantum numbers in (38) can be evaluated with the aid of the Wigner-Eckhart theorem

$$(I_t M_t | T_{LM} | I_1 M_1) = C^{I_1\ L\ I_t}_{M_1\ M\ M_t} (I_t || T_L || I_1), \tag{39}$$

where $(I_t||T_L||I_1)$ denotes the usual reduced matrix element independent of magnetic quantum numbers. Substituting (2) into (38) gives

$$\mathscr{W}_e = \frac{8\pi^2\alpha}{L(L+1)(2L+1)} k^2 \frac{2I_t+1}{2I_1+1} \sum_{\kappa_t} C_{\kappa_t, \kappa_1} \times \left| R_{\kappa_t}(e)(\mathrm{f}||\mathbf{j}_n \cdot \mathbf{A}_L^*||\mathrm{i}) + (\mathrm{f}||\mathbf{j}_n \cdot \nabla \Phi Y_L^*||\mathrm{i}) + (\mathrm{f}||\mathbf{j}_n \cdot \frac{\hat{\mathbf{r}}}{\gamma} \Theta Y_L^*||\mathrm{i}) \right|^2. \tag{40}$$

The remaining sum in this expression is taken over all the possible final electron states listed in table 1, and C_{κ_f, κ_1} is the weighting factor tabulated by Rose [6]). As in eq. (2), the first term in the sum in (40) is the gamma-ray contribution to the conversion process, while the remaining two terms are the two types of penetration terms.

In the penetration-conversion-limit, the gamma-ray terms are neglected relative to one or both of the penetration terms. For the L_I and L_{II} conversion of the 84.2 keV transition in Pa^{231}, we 1) neglect the gamma-ray contribution in the $s_{\frac{1}{2}} \leftrightarrow p_{\frac{1}{2}}$ electron transmissions, and 2) neglect both gamma-ray and penetration contributions for $s_{\frac{1}{2}} \rightarrow p_{\frac{3}{2}}$ and $p_{\frac{1}{2}} \rightarrow d_{\frac{3}{2}}$ transitions. The neglect of the penetration terms in the latter two electron transitions † is justified by the normal L_{III} conversion. In this case, the result is

$$\mathcal{W}_e(L_I, L_{II}) \rightarrow \frac{8\pi^2\alpha}{3} k^2 \frac{2I_f+1}{2I_1+1} \Big| (f||\mathbf{j}_n \cdot \nabla \Phi Y_1^*||i) + (f||\mathbf{j}_n \cdot \frac{\hat{\mathbf{r}}}{r} \Theta Y_1^*|i) \Big|^2 \frac{m_e c^2}{\hbar} \sec^{-1}, \quad (41)$$

having made use of the fact that $C_{+1,-1} = C_{-1,+1} = 2$.

Substituting the appropriate expressions for Φ and Θ from table 5 into (41), the asymptotic expressions for the absolute rates of L_I and L_{II} electron conversion are obtained for the 84.2-keV E1 transition in Pa^{231} which have been quoted in the text (eqs. (18), 19)). In evaluating these we have used the spin assignments $I_f = \frac{3}{2}$; $I_1 = \frac{5}{2}$ given in the literature [3, 13]).

Appendix 4

KNITTING-NEEDLE NUCLEUS MATRIX ELEMENTS

The leading gamma-ray matrix element and the $\mathbf{j} \cdot \nabla$ type of penetration matrix elements may be put in the general form

$$\int \mathbf{j}_n \cdot \nabla r^p Y_{1M}^* = -ik \int \phi_f^* r^p Y_{1M}^* \phi_1 \quad (42)$$

by use of the continuity equation (5). The asymptotics election rules may now be found by inspection. For $p = 1$ the gamma-ray matrix element is obtained, and its selection rules are given in table 6.

For $p = 3$, the leading $\mathbf{j} \cdot \nabla$ type of penetration matrix element is obtained. Whenever the gamma-ray matrix element is allowed, so is this matrix element. However, the converse is not true, and the penetration matrix element may be allowed while the gamma-ray is forbidden. These additional possibilities are listed in table 7. The uninteresting transitions which involve ΔN, $\Delta n = \pm 3$ have been deliberately omitted in this table.

† See the second footnote of sect. 6.

Another set of selection rules may be written down for the next $\mathbf{j}\cdot\nabla$ type of penetration matrix element, $p = 5$. These will include all the cases already listed in tables 6 and 7, plus additional possibilities which are, however, of no immediate interest here.

TABLE 6

Asymptotic selection rules for the E1 gamma-ray matrix element $r^1Y_1^*$

ΔK	Operator	ΔN	Δn	$\Delta\Lambda$	$\Delta\Sigma$
$+1$	$x+iy$	±1	0	$+1$	0
0	z		±1	0	
-1	$x-iy$		0	-1	

TABLE 7

Asymptotic selection rules for the penetration matrix element $r^3Y_1^*$

ΔK	Operator	ΔN	Δn	$\Delta\Lambda$	$\Delta\Sigma$
$+1$	$z^2(x+iy)$	±1	±2	$+1$	0
0	$z(x^2+y^2)$		∓1	0	
-1	$z^2(x-iy)$		±2	-1	

These are identical with those for the penetration matrix element $(\mathbf{j}_n\cdot\hat{\mathbf{r}}/r)r^3Y_1^*$, if $\mathbf{j}_n$ is the convection-current term in (10). For simplicity, only those cases are shown which are different from those for the gamma-ray matrix element (table 6), and the possibilities involving ΔN, $\Delta n = \pm3$ are omitted.

The selection rules for the $\mathbf{j}\cdot\hat{\mathbf{r}}$ type of penetration matrix element depend on the form of the nuclear current. For example, for the convection current term in (10), we find

$$\int \mathbf{j}_{\text{conv.}}\cdot\frac{\hat{\mathbf{r}}}{r}r^qY_{1M}^* = \frac{e}{2iM}\left[2\int\phi_f^*\,Y_{1M}^*r^{q-1}\frac{\partial}{\partial r}\phi_1+(q+1)\int\phi_f^*\,Y_{1M}^*r^{q-2}\phi_1\right]. \quad (43)$$

Of the two terms in (43), only the first need now be considered, since the second is of the form (42) already discussed and does not lead to new possibilities. For the lowest case of interest, $q = 3$, the first term in (43) leads to selection rules which are identical with those already given in tables 6 and 7, and all of these remarks made for matrix elements of $r^3Y_{1M}^*$ also apply here. The second term in (43) is identical with the gamma-ray matrix element in this case, with the selection rules already given in table 6.

For the spin current term in (10) we have

$$\int \mathbf{j}_{\text{spin}} \cdot \frac{\hat{\mathbf{r}}}{r} r^q Y_{1M}^* = \mu \sqrt{\frac{3}{4\pi}}\, \hat{\mathbf{e}}_M^* \cdot \int \phi_f^* \boldsymbol{\sigma} \times \mathbf{r} \phi_i r^{q-3}. \tag{44}$$

For the lowest case of interest, $q = 3$, this spin matrix element reproduces all the gamma-ray selection rules given in table 6 plus additional possibilities which involve spin flips. These extra possibilities are given in table 8.

TABLE 8

Asymptotic selection rules for the penetration matrix element $(\mathbf{j}_n \cdot \hat{\mathbf{r}}/r)r^3 Y_1^*$, where $\mathbf{j}_n$ is the spin current term in (10).

ΔK	Operator	ΔN	Δn	$\Delta \Lambda$	$\Delta \Sigma$
$+1$	$z\sigma_{+1}$	± 1	± 1	0	$+1$
0	$(x-iy)\sigma_{+1}$	± 1	0	-1	$+1$
0	$(x+iy)\sigma_{-1}$	± 1	0	$+1$	-1
-1	$z\sigma_{-1}$	± 1	± 1	0	-1

For simplicity, only those cases are shown which are different from those for the gamma-ray matrix element (table 6). The analogous results for the convection current are given in table 7.

For $q = 5$, the spin matrix element reproduces all the results of tables 6 and 8 along with additional possibilities which have not appeared before. These involve $\Delta n = \pm 2$, among others, and are given in table 9. Again, cases with ΔN, $\Delta n = \pm 3$ have been deliberately omitted. Although these higher spin-current selection rules in table 9 have not been employed in the present discussion of Pa^{231}, they may be important in explaining the penetration effects in other cases, e.g. odd-neutron Np^{237}.

TABLE 9

Asymptotic selection rules for the penetration matrix element $(\mathbf{j}_n \cdot \hat{\mathbf{r}}/r)r^5 Y_1^*$, where $\mathbf{j}_n$ is the spin current in (10)

K	Operator	ΔN	Δn	$\Delta \Lambda$	$\Delta \Sigma$
$+1$	$(x^2+y^2)z\sigma_{+1}$	± 1	∓ 1	0	$+1$
$+1$	$(x+iy)z^2\sigma_0$	± 1	± 2	$+1$	0
0	$(x+iy)z^2\sigma_{-1}$	± 1	± 2	$+1$	-1
0	$(x-iy)z^2\sigma_{+1}$	± 1	± 2	-1	$+1$
-1	$(x-iy)z^2\sigma_0$	± 1	± 2	-1	0
-1	$(x^2+y^2)z\sigma_{-1}$	± 1	∓ 1	0	-1

For simplicity, only those cases are shown which are different from those already listed in tables 6 and 8, and the possibilities involving ΔN, $\Delta n = \pm 3$ are omitted.

In using a more realistic nuclear model for the evaluation of the various penetration matrix elements, one must be careful to use the form of the current which satisfies the continuity eq. (5). For example, in the more general Nilsson model [10, 13]), which uses $\mathbf{l} \cdot \mathbf{s}$ and $\mathbf{l} \cdot \mathbf{l}$ terms in its Hamiltonian, as well as a truncated diagonalization, current contributions arising from these three velocity dependent sources must be included in $\mathbf{j}_n$ for consistency. The presence of these additional terms would be directly manifest in the evaluation of the $\mathbf{j} \cdot \hat{\mathbf{r}}$ type of electric penetration matrix elements, as well as in all magnetic matrix elements. Thus, as well as being important in electric penetration effects, these terms could also appear significantly in M1 transitions and in static magnetic-dipole moments.

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INTERACTION THEORY IN A NEW ISOBARIC SPIN SPACE

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Abstract: Utiyama's theory of interactions is extended to an isobaric spin space invariant under the group R_3^* of three-dimensional complex rotations. Utiyama's formalism rests on the utilisation of certain simple invariance groups of the Lagrangian, which turn out to be R_2 and R_3 (two-dimensional and three-dimensional real rotation groups). These groups are found to be determined by the invariance properties of the Hilbert space built out of the representations of R_3^*, in which we have defined our basic "elementary" fields. The group R_2 is associated with charge conservation, while R_3 yields the conservation of isobaric spin and strangeness. The conservation of baryon number is automatically given by the special form of the baryon fields. The group R_2 introduces the electromagnetic field; R_3 introduces "complexes" of pion-fields and one justifies the $\Delta I = \frac{1}{2}$ selection rule. The theory shows moreover that one can only have strangeness singlets which eliminate all fields belonging to the $D(0, 1)$, $D(1, \frac{1}{2})$ and $D(\frac{1}{2}, 1)$ representations of R_3^* which have not been observed experimentally. We can also explain the difference between Σ^0 and Λ^0 in our new space.

1. Introduction

In a preceding paper [1]), we have introduced a new isobaric spin space invariant under the group R_3^* of three-dimensional complex rotations (locally isomorphic to the Lorentz group) deduced from a model of extended particles in Minkowski's space-time. The object of the present paper is to present the results obtained (detailed calculations will be published elsewhere) [2]), when we extend into this space the formalism developed by Utiyama [3]) to calculate interactions for Lagrangians invariant under the Lorentz group $\mathscr{L}_4$.

As we shall see, the discussion made here on the invariance groups of the Lagrangians solves certain difficulties of the particle classification [1]) associated with this space, such as the Λ^0 problem.

Let us first briefly recall Utiyama's essential results. Denoting by Q a system of fields, by $Q_{,\mu}$ their derivatives $\partial_\mu Q$, we can introduce the action integral

$$I = \int_\Omega \mathscr{L}(Q, Q_{,\mu}) \mathrm{d}^4x, \tag{1}$$

which is invariant under the infinitesimal transformation of a Lie group G (with n parameters ε^a, $a = 1, 2, \ldots, n$), i.e.

$$Q \to Q + \delta Q, \qquad \delta Q = T_a Q \varepsilon^a. \tag{2}$$

The quantities T_a are the transformation matrices of the Q fields which satisfy

$$[T_a, T_b] = f_a{}^c{}_b T_c, \tag{3}$$

$f_a{}^c{}_b = -f_b{}^c{}_a$ being the "structure constants" of G.

If one replaces the parameters ε^a by n arbitrary functions $\varepsilon^a(X_\mu)$ that is, if one assumes the invariance of the system under the extended group G' defined by $\varepsilon^a(X_\mu)$, one can show that this invariance is only possible if:

a) one introduces as many interaction fields, represented by four vectors $A^a_\mu(X)$ (μ denoting the usual tensor components), as the number a of parameters of the group G;

b) the interaction between A and Q takes the form $A^a_\mu j^a_\mu$, where j^a_μ are currents deduced in a simple way from the initial Lagrangian;

c) the fields $A^a_\mu(X)$ transform according to the law

$$\delta A^a_\mu = f_b{}^a{}_c A^b_\mu \varepsilon^c(X_\mu) + \partial_\mu \varepsilon^a(X_\mu); \tag{4}$$

d) the new Lagrangian $\mathscr{L}'(Q, A)$ is obtained by replacing, in the original $\mathscr{L}$, terms $\partial_\mu = \partial/\partial X_\mu$ by

$$\nabla_\mu = \partial_\mu - T_a A^a_\mu; \tag{5}$$

e) the field equations for A^a_μ are deduced from the Lagrangian $L_0(F^a_{\mu\nu})$ satisfying the condition

$$\tfrac{1}{2} \frac{\partial L_0}{\partial F^a_{\mu\nu}} f_c{}^a{}_b F^b_{\mu\nu} = 0, \tag{6}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \tfrac{1}{2} f_b{}^a{}_c (A^b_\mu A^c_\nu - A^b_\nu A^c_\mu). \tag{7}$$

The main point of our paper is this: if one can obtain elementary particle fields in isobaric space, with the help of a Lagrangian formalism invariant under a group R_3^* (which plays therein the role attributed by Utiyama to $\mathscr{L}_4$), there is no reason (if these "internal" Lagrangians are also invariant under Lie groups G) why one could not extend Utiyama's formalism into isobaric spin space and characterize in this way the "internal" nature of fields appearing in certain types of interactions †. This dictates the plan of our paper. In the next section, we give general results characterizing the extension of Utiyama's formalism to our new isobaric spin space invariant under R_3^*. In the last section, we first discuss the invariance groups and then apply this formalism to the study of typical interactions in this space.

In all this study, the Greek indices vary from 1 to 4, the Latin indices from 1 to 3 and we utilize the usual summation convention on indices repeated twice. In ref. [2]), the interested reader will find the detailed calculations which lead to the general results of the next section with an application in which the groups G are not bound to the

† According to Utiyama's terminology, we call "interactions of the first class" a family of the interactions derived by its formalism, and "interactions of the second class" the others.

invariance properties of the Hilbert subspaces associated with the irreducible representations of R_3^*, so that this application has therefore only a mathematical interest.

2. Extension of Utiyama's Formalism

2.1. PROPERTIES OF THE GROUP R_3^*

According to our program, let us first briefly recall the essential properties [4]) of the R_3^* group, which can be summarized as follows:

a) As a group, R_3^* is locally isomorphic to the orthochronous proper Lorentz group $\mathscr{L}_4^+$. However, while $\mathscr{L}_4^+$ considered as transformation group, acts on four-dimensional space-time, R_3^* operates on its manifold. The group R_3^* is the complexified of the three-dimensional rotation group R_3 in the sense that the three real Euler angles $\omega = \{\theta, \varphi, \psi\}$ are replaced by three complex angles ω^+ and their complex conjugates ω^-. This complexification procedure (also introduced by Cartan [5])) is not the procedure usually utilized in the theory of Lie groups.

b) The differential operators of the corresponding Lie algebra satisfy the relations

$$[J_i^\pm, J_k^\pm] = -\varepsilon_{ijk} J_j^\pm, \tag{8}$$

$$[J_k^+, J_i^-] = 0, \qquad [J_3^\pm, J_k^\pm J_k^\pm] = 0. \tag{9}$$

c) The finite-dimensional representations of R_3^* are compact, if one introduces two types of complex quantities: the first type such that ω^+ and ω^- are constructed with the imaginary unit $i = \sqrt{-1}$ associated with the metric of space-time, the second type such that the eigenfunctions of the quantum operators $J_i'^{\pm} = -j\hbar J_i^\pm$ are constructed with the imaginary unit $j = \sqrt{-1}$ connected with the quantization procedure. The finite-dimensional representations $\mathscr{D}(l^+, l^-)$ are then characterized by two positive numbers l^+ and l^- which take independently integer or half integer values.

d) The parity operation P is an involutive automorphism, the operations C (charge conjugation) and T (time reversal) being two involutions. The CTP theorem then corresponds to the fact that an automorphism is the product of two involutions. The operation P transforms any function of ω^+ into a function of ω^- and vice versa. The group R_3^* with its operator domain has two types of inequivalent representations, namely $\mathscr{D}(l^+, l^-)$ and $\mathscr{D}(l^-, l^+)$ which can be deduced from one another by the operation P.

2.2. CASE OF CLASSICAL FIELDS

To extend Utiyama's formalism, we consider a system of fields $Q(\omega^+, \omega^-)$ and assume the existence of an action integral

$$I = \int_\Omega \mathscr{L}(Q, J_k^+ Q, J_k^- Q) \mathrm{d}\omega^+ \mathrm{d}\omega^-, \tag{10}$$

where $\mathrm{d}\omega^+ \mathrm{d}\omega^-$ denotes the volume element in the space of parameters ω^+, ω^- and Ω is an arbitrary volume in that space. We also assume that I is invariant under the

infinitesimal transformations of an n-parameter Lie group G, with parameters $\varepsilon^a (a = 1, 2, \ldots n)$. Supposing then that this invariance is only local on the manifold of R_3^* (i.e., replacing the n parameters ε^a by n arbitrary functions of ω^+ and ω^-), one can (following Utiyama's steps) look for the conditions under which we can maintain the invariance of I under this extended group G'. A first difference from Utiyama appears here. Because of the involutive automorphic character of P, we have two possibilities, corresponding to functions $\varepsilon^a(\omega^+)$ or $\varepsilon^a(\omega^-)$. We shall discuss, in the next section, the physical meaning of these two possibilities. However, as they lead to the same expressions, we shall give the results corresponding to the operation $\varepsilon^a \to \varepsilon^a(\omega)$, where ω represents ω^+ or ω^-. In the same way, we shall use J_k for J_k^+ and J_k^- when the result will not depend on the index $\pm$.

Another difference from Utiyama's original formalism is evidently related to the structure eqs. (8) of R_3^* which replace the commutation relations $[\partial_\mu, \partial_\nu] = 0$.

However, if one follows step by step Utiyama's reasoning, one can show that the invariance of I under the extended group G' is only possible if:

a) we introduce as many interaction fields, represented by self-dual antisymmetric tensors $A_k^a(\omega)$ as the number a of parameters of the group G;

b) the interaction between the fields Q and A takes the form $j_k^a A_k^a$, where j_k^a denote the currents easily deduced from $\mathscr{L}$;

c) the fields $A_k^a(\omega)$ transform under G' according to the law

$$\partial A_k^a = f_b{}^a{}_c A_k^b \varepsilon^c(\omega) + J_k \varepsilon^a(\omega); \tag{11}$$

d) the new Lagrangian $\mathscr{L}'(Q, A)$ is obtained by substituting in the original $\mathscr{L}$ the operators J_k by

$$\nabla_k = J_k - T_a A_k^a; \tag{12}$$

e) the field equations for A_k^a are deduced from the Lagrangian $L_0(F_{ij}^a)$ satisfying the condition

$$\tfrac{1}{2} \frac{\partial L_0}{\partial F_{ij}^a} f_c{}^a{}_b F_{ij}^b = 0, \tag{13}$$

with

$$F_{ij}^a = J_i A_j^a - J_j A_i^a - \tfrac{1}{4} f_b{}^a{}_c (A_i^b A_j^c - A_j^b A_i^c) + \varepsilon_{ijk} A_k^a, \qquad \delta F_{ij}^a = f_b{}^a{}_c F_{ij}^b \varepsilon^c. \tag{14}$$

As stated before, detailed calculations are given in ref. [2]).

2.3. CASE OF QUANTIZED FIELDS

If we now substitute in the expression (10) a quantized field for the Q field (i.e. spinors or tensors belonging to representations $\mathscr{D}(l^+, l^-)$ and $\mathscr{D}(l^-, l^+)$ of R_3^* for Q, and quantized operators $J_k'^{\pm} = -j\hbar J_k^{\pm}$ for J_k^+) the preceding theory yields an extension of Utiyama's formalism to the case of quantized fields. One then sees

immediately that we must substitute

$$[J_i'^{\pm}, J_j'^{\pm}] = -j\hbar\varepsilon_{ijk}J_k'^{\pm} \tag{15}$$

for relations (8).

As shown previously [1]), the matrix elements of the representation $\mathscr{D}(l^+, l^-)$ are functions $Z^{m^+,m^-,m'}_{l^+,l^-,s'}(\omega^+, \omega^-)$, which can be written explicitly as

$$\begin{aligned} &Z^{m^+,m^-,m'}_{l^+,l^-,s'}(\omega^+, \omega^-) \\ &\quad = \sum_{-m'^+,\,-m'^-} (l^+, l^-, s', -m'|l^+, l^-, -m'^+, -m'^-) Y^{m^+,m'^+}_{l^+}(\omega^+) Y^{m^-,m'^-}_{l^-}(\omega^-), \\ &Y^{m,m'}_{l}(\omega) \\ &\quad = (\sin\tfrac{1}{2}\theta)^{-m+m'}(\cos\tfrac{1}{2}\theta)^{-m-m'} e^{j(m\varphi+m'\psi)} \frac{d^{j-m}}{d(\sin^2\tfrac{1}{2}\theta)^{j-m}}[(\sin\tfrac{1}{2}\theta)^{2j-2m'}(\cos\tfrac{1}{2}\theta)^{2j+2m'}], \end{aligned} \tag{16}$$

where $(l^+, l^-, s', -m'|l^+, l^-, -m'^+, -m'^-)$ are Clebsch-Gordan parameters.

As a consequence, the rows and columns are ordered with the help of two numbers sets $\{s', m'\}$ and $\{m^+, m^-\}$ each row belonging to a specific Hilbert subspace, so that we can classify different state vectors of these subspaces with the help of their eigenvalues for certain internal operators. One can show that in this way one obtains a classification very similar to the Nishijima-Gell-Mann scheme [1]). The above-mentioned internal operators can thus be associated with isobaric spin, strangeness and baryon number operators. One recalls that m^+, m^-, s' and m' can take the following values:

$$\begin{aligned} m^{\pm} &= -l^{\pm}, -l^{\pm}+1, \ldots, (l^{\pm}-1), l^{\pm}, \\ s' &= l^+ + l^-, l^+ + l^- - 1, \ldots, |l^+ - l^-|, \\ m' &= -s', -s'+1, \ldots, s'-1, s'. \end{aligned} \tag{17}$$

Applying now these considerations to the quantized fields A^a_k we see, as a consequence of (a) of 2.2, that the new interaction fields must belong (if we consider them as elementary particles) to the representations $\mathscr{D}(1, 0)$ or $\mathscr{D}(0, 1)$ of R_3^*.

For these representations, s' only takes the value 1 so that we have three Hilbert subspaces corresponding to $m' = 1, 0, -1$. As every one of these is three-dimensional, we have nine antisymmetric self-dual tensors in $\mathscr{D}(1, 0)$ (and nine in $\mathscr{D}(0, 1)$) which we can classify as we said before, by their eigenvalues with respect to isobaric spin, strangeness and baryon number operators [2]). Among these, all are not possible if we limit ourselves to charge 1, 0, -1. One sees in particular that the π-mesons form an isobaric spin triplet in $\mathscr{D}(1, 0)$ with $m' = 0$.

Moreover, we know [1]) that the antisymmetric self-dual tensors A^a_k are solutions of the second order differential equations

$$[J'_k J'_k - l(l+1)\hbar^2]A^a_k = 0 \qquad \text{with } l = 1. \tag{18}$$

We shall then have to check that eq. (18) can be deduced from Lagrangians $L_0(F^a_{ij})$ satisfying (13) and will see that this is not always possible. In the rest of this paper we shall only consider the extension to quantized fields of Utiyama's formalism.

3. Application to Internal Interaction Theory

3.1. INVARIANCE GROUPS IN THE THREE TYPES OF INTERACTIONS

In order to examine with Utiyama's formalism the laws of conservation of charge, baryon number, isobaric spin and strangeness in electromagnetic, weak and strong interactions, we must evidently determine the different groups G which characterize the corresponding interactions, but the following discussion remains valid for the interactions of the second class (for which Utiyama's formalism cannot be used).

In usual quantum theory the introduction of these groups is purely formal. We shall now see that this is not the case in our isobaric spin space.

Let us start from nucleons. We have shown [6]) that the baryon wave fields have the variance of antisymmetric self-dual tensors, each component being a two-spinor. For nucleons, two of these components vanish, so that we can write their wave functions in the form

$$\Phi_n = \begin{pmatrix} Z^{-\frac{1}{2},\,0,\,\frac{1}{2}}_{\frac{1}{2},\,1,\,\frac{1}{2}}(\omega^+, \omega^-) \\ Z^{\frac{1}{2},\,0,\,\frac{1}{2}}_{\frac{1}{2},\,1,\,\frac{1}{2}}(\omega^+, \omega^-) \end{pmatrix}, \tag{19a}$$

$$\Phi_p = \begin{pmatrix} Z^{\frac{1}{2},\,0,\,\frac{1}{2}}_{\frac{1}{2},\,1,\,\frac{1}{2}}(\omega^+, \omega^-) \\ Z^{-\frac{1}{2},\,0,\,\frac{1}{2}}_{\frac{1}{2},\,1,\,\frac{1}{2}}(\omega^+, \omega^-) \end{pmatrix}. \tag{19b}$$

Neglecting normalisation factors, these expressions become with the help of (16)

$$\Phi_n = aY_1^{0,\,0}(\omega^-) \begin{pmatrix} Y_{\frac{1}{2}}^{-\frac{1}{2},\,\frac{1}{2}}(\omega^+) \\ Y_{\frac{1}{2}}^{\frac{1}{2},\,\frac{1}{2}}(\omega^+) \end{pmatrix} + bY_1^{0,\,1}(\omega^-) \begin{pmatrix} Y_{\frac{1}{2}}^{-\frac{1}{2},\,-\frac{1}{2}}(\omega^+) \\ Y_{\frac{1}{2}}^{\frac{1}{2},\,-\frac{1}{2}}(\omega^+) \end{pmatrix}, \tag{20a}$$

$$\Phi_p = aY_1^{0,\,0}(\omega^-) \begin{pmatrix} Y_{\frac{1}{2}}^{\frac{1}{2},\,\frac{1}{2}}(\omega^+) \\ Y_{\frac{1}{2}}^{-\frac{1}{2},\,\frac{1}{2}}(\omega^+) \end{pmatrix} + bY_1^{0,\,1}(\omega^-) \begin{pmatrix} Y_{\frac{1}{2}}^{\frac{1}{2},\,-\frac{1}{2}}(\omega^+) \\ Y_{\frac{1}{2}}^{-\frac{1}{2},\,-\frac{1}{2}}(\omega^+) \end{pmatrix}, \tag{20b}$$

with $a = (\frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2} | \frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}) = 1/\sqrt{3}$ and $b = (\frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2} | \frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}) = 1/\sqrt{3}$, so that one passes from Φ_p to Φ_n by an infinitesimal transformation of the rotation group. More precisely we have

$$\Phi_p = \sigma_1 \Phi_n, \qquad \Phi_n = \sigma_1 \Phi_p, \qquad \sigma_2 \Phi_p = i\sigma_3 \Phi_n, \tag{21}$$

where σ_i are the Pauli matrices. In particular, if one changed the sign of one of the components of Φ_n or Φ_p this would be equivalent to their multiplication by one of the matrices σ_i, i.e. a physical equivalence. We shall see later the importance of this remark.

As Φ_n and Φ_p are linear combinations of spinors, belonging to the representation $\mathscr{D}(\frac{1}{2}, 0)$ of R_3^*, we can build with them an isomorphism with the $\mathscr{D}(\frac{1}{2})$ representation of the rotation group R_3.

The isobaric spin operator being [1]) $\sigma_3 J_3^+$ we see that if we consider the two-component spinors

$$\Phi = \begin{pmatrix} \Phi_p \\ \Phi_n \end{pmatrix},$$

with $\sigma_3 J_3^+ \Phi_n = -\frac{1}{2}\Phi_n$ and $\sigma_3 J_3^+ \Phi_p = \frac{1}{2}\Phi_p$, we have

$$(\tau_3 \otimes \sigma_3) J_3^+ \Phi = \tfrac{1}{2}\Phi,$$

where the symbol $\otimes$ denotes the direct product and τ_3 is a Pauli matrix but corresponding to an infinitesimal transformation on isobaric spin.

Relations (21) and (22) show that the isobaric spin of nucleons is bound to the representation $\mathscr{D}(\frac{1}{2})$ of the three-dimensional rotation group R_3 (thus denoted to distinguish it from the usual SO(3), since R_3 operates on its own manifold as a consequence of the preceding isomorphism.

This result can be generalized to all fields if we notice that we have

$$\mathscr{D}'(l^+, l^-) = \mathscr{D}(l^+, 0) \otimes \mathscr{D}(0, l^-) \tag{23}$$

and it is possible to put into an isomorphic correspondence $\mathscr{D}(l^+, 0)$ and the $\mathscr{D}(l)$ representation of R_3 with $l = l^+$.

In fact, we find only singlets, doublets or triplets of isobaric spin, because we can only use the $\mathscr{D}(0)$, $\mathscr{D}(\frac{1}{2})$ or $\mathscr{D}(1)$ representations of the G-group R_3. As a consequence, we can say that in the representations of R_3^*, we can find subgroups, isomorphic to the representations of R_3, which yield, inside each particle family, subfamilies of isobaric spin multiplets. The conservation of isobaric spin is obtained if we impose on the Lagrangian invariance under R_3 group.

One can ask if it is not possible to obtain in the same way strangeness subfamilies. This is not possible because of (23) and of the form $\sigma_3 J_3^-$ of the strangeness operator [1]). If it were, we should have to utilize once more representations of the R_3 group. But we have shown [4]) that this would give

$$R_3^* = R_3 \otimes R_3, \tag{24}$$

which is not exact. The Lagrangian being already invariant under R_3^* and R_3 can only have the supplementary invariance under a group isomorphic to the gauge group R_2. One therefore concludes that a) there exist only strangeness singlets (because R_2 is abelian), and b) for interacting fields which have isobaric spin and strangeness the invariance of the Lagrangian under the G-group R_3 ensures the simultaneous conservation of isobaric spin and of strangeness (since the Lagrangian is then also invariant under R_2).

The fact that we can only have strangeness singlets has important consequences for the classification of elementary particles associated to R_3^*. Indeed, we can see that it solves still open problems in our scheme [1]). In particular we can say that:

(i) π-mesons have no corresponding particles in $\mathscr{D}(0, 1)$ for they would correspond to a strangeness triplet.

(ii) All wave fields belonging to $\mathscr{D}(\frac{1}{2}, 1)$ or $\mathscr{D}(1, \frac{1}{2})$ are not possible, for this would also lead to the possible appearance of strangeness multiplets (one can check immediately that this implies the suppression in our scheme of all unobserved baryonic particles).

(iii) This also leads to a possible solution of the Λ^0 problem, which we had left open in our paper. Indeed we recall that for Σ^0, as for nucleons, two of the tensor components vanish and the third can be written as

$$A_3 = \begin{pmatrix} Z^{0,\,-\frac{1}{2},\,\frac{1}{2}}_{1,\,\frac{1}{2},\,\frac{1}{2}}(\omega^+, \omega^-) \\ Z^{0,\,\frac{1}{2},\,\frac{1}{2}}_{1,\,\frac{1}{2},\,\frac{1}{2}}(\omega^+, \omega^-) \end{pmatrix}.$$

But here, if we change the sign of one of the components, we obtain a new internal wave field with

$$A'_3 = \begin{pmatrix} Z^{0,\,-\frac{1}{2},\,\frac{1}{2}}_{1,\,\frac{1}{2},\,\frac{1}{2}}(\omega^+, \omega^-) \\ -Z^{0,\,\frac{1}{2},\,\frac{1}{2}}_{1,\,\frac{1}{2},\,\frac{1}{2}}(\omega^+, \omega^-) \end{pmatrix},$$

which is not equivalent to A_3 because (in contrast to the nucleon case) we cannot pass from A_3 to A'_3 by an infinitesimal transformation of R_3, since there is no correspondence between the representations $\mathscr{D}(0, l-)$ of R_3^* and the representations of $\mathscr{D}(l)$ R_3. We can thus tentatively represent Λ^0 by the tensor

$$A^{\Lambda_0} = (A_1 = 0,\ A_2 = 0,\ A'_3),$$

which has the same isobaric spin, strangeness and baryon number values as the A^{Σ_0} tensor.

To conclude this section, let us finally study the group corresponding to charge conservation. We have shown [6]) that the Pauli gauge group corresponds to one degree of freedom in the determination of the Euler angles ψ^+ and ψ^- which appear in the functions $Z^{m^+;\,m^-;\,m'}_{l^+,\,l^-,\,s'}(\omega^+, \omega^-)$. This shows evidently that the Pauli group is isomorphic to the plane rotation group R_2.

We can now establish the following result: the conservation of baryon number is a necessary immediate consequence of the invariance of the Lagrangian under R_3^* and of the specific form of the "internal" baryon wave functions.

Indeed the scalar character (under R_3^*) of the Lagrangian implies that terms containing baryon wave fields must be bilinear, for it is impossible to obtain scalar forms by the product of a lepton field (spinor) and a baryon field represented by a six-component geometrical entity, i.e. an antisymmetric self-dual tensor.

Thus, each bilinear form is either purely baryonic or purely leptonic. This joined to charge conservation expresses the separate conservation of baryon and lepton numbers. Indeed, in our theory, to an arbitrary rotation of angles $\psi^\pm$ (charge conservation) corresponds the operator S'_3, which we have identified with the baryon number operator $\mathscr{N}$ ($\mathscr{N} = 2S'_3$) because it has the same eigenvalues. As a consequence, charge conservation, implying that of S'_3, makes it possible to write the baryon number conservation under the usual form

$$N = N^+ - N^- = \text{const.},$$

N^+ and N^- being the total number of baryons and antibaryons in any reaction.

This explains why we have only three independent conservation laws. The conserva-

tion of baryon number being a consequence of the invariance of the theory under R_3^*, we then have the following scheme:

invariance group	conserved quantities	interaction field
R_2	charge	A_k
R_3	spin and strangeness	$A_k^a \quad (a = 1, 2, 3)$

As we said before, these results about the conservation laws are valid whatever class of interactions one considers. The interest of the Utiyama's formalism, to which we shall limit our present study, is to furnish immediately interaction Lagrangians while, in those cases where it is not applicable, it is necessary to construct all Lagrangians scalar under R_3^* and, at the same time, invariant either under R_2 or under R_2 and R_3.

3.2. APPLICATION OF UTIYAMA'S FORMALISM

As shown in sect. 2, one of the main differences between Utiyama's original formalism and its extensionto our isobaric spin space is the possibility of considering functions $\varepsilon^a(\omega^+)$ and $\varepsilon^a(\omega^-)$. After the discussion of subsect. 3.1, we can eliminate the second possibility (which would lead to antisymmetric anti-dual tensors $A_k(\omega^-)$ belonging to $\mathscr{D}(0, 1)$ and forming strangeness multiplets, and restrict ourselves to the first case, i.e. functions $\varepsilon^a(\omega^+)$ which yield interaction fields belonging to $\mathscr{D}(1, 0)$.

Let us now consider the group R_2 and the corresponding interaction fields. The problem is to discover antisymmetric self-dual tensors which belong to $\mathscr{D}(1, 0)$ and satisfy relations (18) which should be deduced from the Lagrangian $L_0(F_{ij})$. This will lead to an important limitation of possible Lagrangians.

If we introduce the vectors $G_k = \frac{1}{2}\varepsilon_{ijk}F_{ij}$, the Lagrangian G_kG_k leads to the relations

$$G_i - j\varepsilon_{ijk}J'_jG_k = 0. \tag{25}$$

Since we have

$$G_k = \varepsilon_{ijk}J'_iA_j - jA_k,$$

relation (25) can only be identical to (18) if

$$3A_k - J'_k(J'_jA_j) - j\varepsilon_{mlk}J'_lAm = 0, \tag{26}$$

where we have put $\hbar = 1$ and ε_{mlk} is the completely antisymmetrical third rank tensor.

One can then prove that there exist only two neutral fields $A_k(\omega^+)$ which belong to $\mathscr{D}(1, 0)$ and satisfy (26). We can thus assimilate them to an "internal" representation of photons. We shall discuss this point in more detail later.

The explicit form of these fields is

$$
\begin{aligned}
{}^{m'}A_1^{(1)} &= \tfrac{1}{2}a\left(Z_{1,0,1}^{1,0,m'}(\omega^+,\omega^-)+Z_{1,0,1}^{-1,0,m'}(\omega^+,\omega^-)\right),\\
{}^{m'}A_2^{(1)} &= -\tfrac{1}{2}jb\left(Z_{1,0,1}^{1,0,m'}(\omega^+,\omega^-)-Z_{1,0,1}^{-1,0,m'}(\omega^+,\omega^-)\right),\\
{}^{m'}A_3^{(1)} &= cZ_{1,0,1}^{0,0,m'}(\omega^+,\omega^-),
\end{aligned}
\tag{27}
$$

where m' takes the values 1, -1 and $a^2+b^2+c^2=1$.

In this theory the fields $A_k(\omega^+)$ correspond in a sense to the electromagnetic potential in usual quantum theory, the tensors $F_{ij}=J_i'A_j-J_j'A_i-j\varepsilon_{ijk}A_k$ correspond to the electromagnetic field. A simple calculation then shows that the F_{ij} associated with (27) vanish identically.

Let us now consider the case of the R_3 group. The problem is rather complicated as a consequence of the form of the expressions for the corresponding F_{ij}^a, namely,

$$F_{ij}^a = J_i'A_j^a-J_j'A_i^a-j\varepsilon_{ijk}A_k^a+\tfrac{1}{2}\varepsilon^{abc}A_i^bA_j^c.$$

If we introduce as before the vectors $G_k^a=\frac{1}{2}\varepsilon_{ijk}F_{ij}^a$, i.e.

$$G_i^a = \varepsilon_{ijk}J_j'A_k^a-jA_i^a+\tfrac{1}{4}\varepsilon^{abc}\varepsilon_{ijk}A_j^bA_k^c,$$

the Lagrangian is $G_i^aG_i^a$, which yields the field equations

$$G_i^a-j\varepsilon_{ijk}J_j'G_k^a+\tfrac{1}{2}\varepsilon^{acd}\varepsilon_{ilm}A_l^cG_m^d=0. \tag{28}$$

If we compare directly (28) and (18) we obtain an inextricable condition; we shall therefore just establish here that one can find particular solutions. To do this we observe that if the conditions

$$\varepsilon^{abc}\varepsilon_{ijk}A_j^bA_k^c=0, \tag{29}$$

$$\varepsilon_{ijk}J_j'A_k^a=\lambda A_i^a \tag{30}$$

are identically satisfied (λ being a scalar) the field equations become

$$G_i^a-j\varepsilon_{ijk}J_j'G_k^a=0,$$

which implies that we are brought back to the one-parameter group.

The problem is thus to find three fields $A_i^a(\omega^+)$ such that each of them satisfies simultaneously (26), (29) and (30). A solution is evident: the solutions of (27) in which m' takes the three values 1, 0, -1. We thus have three elementary solutions given by the sets of three vectors of which two are equal to zero, the third being one of the vectors (27).

Evidently all linear combinations (in which the coefficients could be constants or functions of ω^-) of the elementary solutions satisfying (29) can be used if they satisfy (29) and we shall now look for such solutions which constitute an isobaric spin triplet (with three charge states) which correspond to the physical properties of the Yang-Mills field [7]). Let us call these solutions "complexes", a complex being a set of three vectors $A_i^a(\omega^+)$ which satisfy (29).

We have established elsewhere [2]) the existence of three such complexes which con-

stitute an isobaric spin triplet with three values of the charge. For more details the interested reader can consult ref. [2]). One sees in particular that $F^a_{ij} = 0$ for such solutions.

Finally we shall examine the question of the internal spin of the fields (27) and of the complexes. We have shown [6]) that if one starts from a scalar Lagrangian $\mathscr{L}(Q(\omega), J_k Q(\omega))$ under R_3^* (where $Q(\omega)$ denotes a system of internal fields) one can establish the existence of tensors M_{ik}, $\mathscr{S}_{ikl}$, Σ_j defined by

$$M_{ik} = \frac{\partial \mathscr{L}}{\partial(J_k Q)} J_i Q - \delta_{ik} \mathscr{L}, \qquad M_{ik} = J_l \mathscr{S}^l_{ik}, \qquad \mathscr{S}^l_{ik} = \tfrac{1}{2}\delta^{ln}_{ik}\Sigma_n \tag{31}$$

(where $\delta^{ln}_{ik} = \delta_{il}\delta_{kn} - \delta_{in}\delta_{kl}$), which correspond in our theory to energy-impulsion, Belinfante-Rosenfeld tensors and spin of the usual quantal theories.

If one utilizes these definitions in the case of the Lagrangians $L_0(F_{ij})$ and $L_0(F^a_{ij})$ one sees immediately that M_{ik} is not symmetrical, so that Σ_l is uniquely defined. One then deduces from the transformation laws of antisymmetrical self-dual tensors (under R_3^*) that the internal spin is unity.

To conclude this section let us make two remarks which apply to the two types of interactions. The first remark is that the selection law for weak interactions

$$\Delta I = \tfrac{1}{2}$$

(where I denotes the amplitude of isobaric spin) results immediately from the form and invariance under R_3 of the interaction terms.

The second remark is that the preceding considerations can in principle be extended to leptons if one considers their internal fields (belonging to the representations $\mathscr{D}(\frac{1}{2}, 0)$ and $\mathscr{D}(0, \frac{1}{2})$ of R_3^*) as isobaric spin singlets bound with the representation $\mathscr{D}(0)$ of R_3.

4. Conclusion

In this paper we have established two different types of results starting first from the isomorphism which exists between certain particle families (a "family" being defined by the representation of R_3^* in which we have defined its internal state vectors) and the representations $\mathscr{D}(\frac{1}{2})$ and $\mathscr{D}(1)$ of R_3 we have shown that to obtain the correct conservation laws we have to assume that strong interactions are invariant under R_3^*, R_3 and R_2, weak interactions being only invariant under R_3 and R_2.

Extending then Utiyama's to "internal" isobaric spin space we have treated a first class of interactions corresponding to the Yang-Mills and electromagnetic external fields. In a subsequent paper we shall treat a second class, namely those which are not susceptible to an Utiyama treatment. This can be done by studying directly all possible non derivative interaction Lagrangians invariant under R_3^*, R_3 and R_2 (strong interactions) or under R_3^* and R_2 only (weak interactions).

We note also that such "internal" types of theories can only give qualitative types of results (such as to determine what types of reactions are possible) and cannot

yield cross-sections, lifetimes etc. The reason is clear: such results can only appear when we have written down total wave fields $\Phi(X, \omega)$ and Lagrangians invariant under R_3^* and $\mathscr{L}_4$. As we shall see in a later paper [8]), however, one cannot associate arbitrarily "internal" with "external" waves because of the local isomorphism of the two groups, so that the problem of external interactions is determined once we have identified and classified elementary particles in isobaric spin space.

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Note added in proof: Introducing in the theory a symmetry operator transforming a right-handed into a left-handed three-dimensional frame, it is possible to use only one type of complex quantities.

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OPTICAL GIANT RESONANCES IN NUCLEAR REACTIONS

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Abstract: In this paper we discuss the physical reasons for optical giant resonances being a very general feature of nuclear reactions. Optical giant resonances, so familiar in the scattering of nucleons on nuclei, have also been observed in elastic cross sections of composite particles scattered by nuclei and in certain reaction channels of a compound nucleus decay. That one should expect this optical giant resonance structure to occur in such a variety of nuclear reactions is mainly attributed, in our arguments, to the effect of the indistinguishability of the nucleons on the interaction between two proximate nuclear clusters. Because of the indistinguishability of the nucleons, the lifetime of the nuclear clusters in a compound nuclear state can often be prolonged to times, of order 10^{-20} to 10^{-22} sec, of sufficient duration compared to the transit times to account for the observed resonances. The relationship of the giant resonance structure and the Coulomb barrier is also discussed.

1. Introduction

In the scattering of nucleons on nuclei, the dependence of averaged elastic and reaction cross sections on bombarding energy and the nuclear radius can be approximately described by the optical model [1]). In this familiar model, the scattering potential has the form

$$V = -(V_0 + iV_1). \tag{1}$$

Essentially, V_0 is responsible for the elastic scattering and V_1 for inelastic scattering. For the occurrence of optical resonance phenomena (giant resonances) and optical interference phenomena in the elastic and reaction total and differential cross sections, it is necessary that the lifetime of the scattering nucleon within the (unexcited) nucleus be of the order of, or larger than, the transit time of the incident nucleon across the nucleus ††. This is equivalent

† Oak Ridge Institute of Nuclear Studies Research Participant, 1960.

†† If the nucleon is a proton, the transit time takes on a special meaning. If the energy of the bombarding proton is large compared to the Coulomb barrier, so that its penetration factor is essentially unity, the transit time is simply the passing time in free flight of the proton across the target nucleus. As the bombarding energy decreases, the transit time refers to the time it requires the proton to tunnel across the target nucleus; this will of course be much larger than the passing time in free flight. In the sequel, the discussions involving charged particles employ this special meaning for transit times.

to saying that the mean free path of the incident nucleon in the nucleus must be of the order of, or larger than, the size of the target nucleus. It is also important that the optical potential in eq. (1) has a refractive edge, i.e. that the shape of the potential undergoes a rather abrupt change at the nuclear surface. When the optical potential does not change abruptly at the nuclear surface, the reflection probability there will be small and the momentum transfer from the nucleus to the scattering nucleon will not have distinct maxima characteristic of resonance phenomena [2]). For example, optical resonance phenomena do not occur for a Gaussian potential.

We briefly summarize the general features of the resonance phenomenon associated with nucleon-nucleus scattering as follows. The probablity per unit time for exciting the nucleus is approximately

$$w_{\mathrm{abs}} \approx \frac{2V_1}{\hbar}, \tag{2}$$

giving the partial energy width

$$\Delta E = 2V_1. \tag{3}$$

For most nuclei, V_0 is about 40—50 MeV and V_1 is about 3—5 MeV (ref. [1])). For nuclei around $A = 200$ and $R = 8$ fm, these values correspond to an energy separation of about 50 MeV between the giant resonances with the same angular momentum and a partial energy width of the resonances, from eq. (3), of about 5—10 MeV. The corresponding lifetime $\tau = \hbar/\Delta E$ is about 10^{-22} sec, a value which is the same order of magnitude as the transit time of the scattering nucleon across the nucleus. Since these times are comparable, the total energy width of the giant resonances is also about 5—10 MeV.

The values of V_0 and V_1 calculated from the two-nucleon interaction [3–5]) are in approximate agreement with the phenomenological values given above. These calculations based on the two-nucleon interaction also indicate that V_0 is rather strongly velocity dependent.

2. Optical Resonances in Nucleus-Nucleus Scattering

We now inquire whether it is reasonable, within the framework of the optical model, to expect resonance phenomena to occur in the scattering of composite particles as α-particles, tritons etc., on nuclei. This inquiry is particularly interesting since certain experiments vividly display such optical phenomena [6]). From a theoretical point of view, these experimental results are, at first, quite unexpected. For example, consider the scattering of α-particles on nuclei. As stated above, for a giant resonance to occur it is necessary that the lifetime of the α-particle in the unexcited nucleus be of the order of or larger than, the transit time across the nucleus. When the α-particle touches the nuclear

surface, the kinetic energy T of each constituent nucleon in the compound nuclear system is about 30 MeV. This follows directly from the Weizsäcker-Bethe formula. Thus we can say that the time required for an exchange of a nucleon in the α-particle with a nucleon in the nucleus is the order of magnitude of

$$\tau \approx \frac{R_\alpha}{\sqrt{\frac{2T}{M_n}}} \approx 2\times 10^{-23}\ \text{sec}. \tag{4}$$

If the time given by eq. (4) were a correct estimate of the lifetime of the α-particle in the nucleus, the latter would be considerably smaller than the transit time of the α-particle through the nucleus and would seeem to preclude the possibility of any optical resonance effects.

Taking the centrifugal barrier effects into account leaves this result essentially unchanged. For example, for an α-particle with an orbital angular momentum $l = 10$ relative to the nucleus, the centrifugal potential is only about 10 MeV for a nucleon at the nuclear surface. Thus eq. (4) is eesentially unchanged and it would still appear that giant resonances are not likely to occur for the scattering of composite particles on nuclei.

In the above discussion, however, an all-important effect has not been considered, the indistinguishability of the nucleons. One cannot simply use eq. (4) to get an estimate of the lifetime of the scattered (composite) particle in the nucleus. The lifetime of the possible nuclear states must be derived from the following kind of (cluster) wave function [7]):

$$\psi = \mathrm{A}\{\varphi_S \varphi_N X(R_S - R_N)\}. \tag{5}$$

In eq. (5), the functions φ_S and φ_N describe the internal degrees of freedom of the bombarding and target nuclei, respectively, while $X(R_S - R_N)$ denotes the dependence on the relative separation of, say, the centre-of-masses of these nuclei. The symbol A is the antisymmetrization operator which acts on *every* pair of nucleons in the wave function (5). This antisymmetrization is required by the Pauli principle and represents an important physical modification of the argument based on eq. (4). For it is now clear that the exchange effects considered in eq. (4), which occur in about 10^{-23} sec, do not determine the lifetime of the nuclear clusters (i.e. the time in wihch the cluster structure (5) is destroyed). Each such exchange of nucleons merely introduces an overall phase factor into the wave function (5) and therefore does not destroy the nuclear cluster wave functions (5) †. The inclusion of exchange effects in (5) often results in a considerable increase of the lifetime of a cluster within nuclear

† The wave function for the case of nucleon — nucleus scattering, considered above, has the form (5) without the antisymmetrization operator A.

matter over the estimation (4)†. If the lifetime of the bombarding cluster in the nucleus is prolonged to the extent that it is about the same as, or larger than, the transit time across the target, then we *would* expect giant optical resonance phenomena to occur.

We can get a rough estimate of the lifetime of a nuclear cluster in a nucleus, i.e., the lifetime of a state given by eq. (5), from the relation

$$\frac{1}{\tau} \approx \frac{A}{V_N} \sigma_{n-c} \sqrt{\frac{2E}{M_n}} r_P r_0 \approx \frac{3}{4\pi} \frac{\sigma_{n-c}}{R_0^3} \sqrt{\frac{2E}{M_n}} r_P r_0. \tag{6}$$

Here, A is the number of nucleons in the target nucleus, which has the volume V_N, and σ_{n-c} is the total cross section for the scattering of a nucleon in the target nucleus *off* the nuclear cluster. (The nuclear cluster is the scattered particle in eq. (5).) The square root factor is the relative velocity. The coefficients r_P and r_0 are reduction factors: r_P accounts for the reduction of final states available to the nucleon, in the nucleon-cluster scattering, because of the Pauli principle and r_0 accounts for the fact that the volume in which the cluster moves may only partially coincide with the volume of the nucleus. The method of computing r_0 is indicated in applications disccussed below. Both r_P and r_0 are less than unity and will, therefore, tend to increase the lifetime of the bombarding nuclear cluster within the nucleus. The calculation of r_P is carried out in essentially the same manner as done by Goldberger [3]), Mittelstaedt [4]) and others, in that the nucleon-cluster scattering is treated as though occurring in a Fermi gas ††.

Eq. (6) is meaningful, of course, only if the bombarding nuclear cluster is not immediately broken up in its collisions with the nucleons of the target nucleus. For example, we would expect eq. (6) to appply when the cluster is energetically stable, as in the case of an α-particle cluster, because of its large binding energy.

When the lifetime of a compound state with the cluster structure (5) is long enough, optical resonance phenomena are expected and an average optical potential of the form (1) can be regarded as sufficient for describing these optical effects. Stated otherwise, when the cluster structure (5) is sufficiently stable, it is reasonable to expect to be able to represent the interaction of the clusters, at least approximately, by means of some complicated velocity dependent two-body potential which may even be complex †††. The situation

† As an extreme example, the low lying energy states of O^{16} have an α-particle cluster structure with an infinite lifetime in first approximation.

†† The numerical evaluation of r_P was carried out at the Oak Ridge National Laboratory on the Oracle.

††† This potential is usually described by a non-diagonal interaction matrix [24]). The imaginary part essentially describes the decay probability and the inelastic scattering probability of the bombarding cluster in the target nucleus.

would be much the same as with ordinary nucleon-nucleus scattering, as commented in the last paragraph of the introduction. The effective potential between the clusters has a depth, corresponding to V_0 in eq. (1), of the order of 50—100 MeV (ref. [7])) and more, and consequently should be largely independent of energy for small bombarding energies of, say, less than 25—30 MeV.

3. Applications

3.1. ELASTIC SCATTERING OF α-PARTICLES AND C^{13}

For bombarding energies of 5—10 MeV, the lifetime determined from eq. (6) is

$$\tau \approx 4\times 10^{-24}/r_p r_0 \text{ sec.} \tag{7}$$

For σ_{n-c} we have taken the value 3.4×10^{-25} cm^2; this is the cross section for n—α scattering for $E_{lab} \approx 50$ MeV (ref. [8])). Taking $r_0 = 0.1$ and $r_p = 0.1-0.2$, the partial energy width determined from (7) would be

$$\Delta E = \frac{\hbar}{\tau} \approx 1—3 \text{ MeV}. \tag{8}$$

The value of r_0 is obtained from the Coulomb energy differences of the $\frac{1}{2}$— mirror levels of O^{17} and F^{17}. The excitation energies of these levels are 3.06 MeV and 3.10 MeV, respectively [9]). These mirror states are taken as the lowest energy states in O^{17} and F^{17} arising from the cluster structure of a C^{13}(gd. st.) and N^{13}(gd. st.) plus an α-particle with relative angular momentum $l = 0$. Thus the Coulomb energy difference of these mirror levels can be used to estimate the volume in which the α-particle cluster moves (see ref. [10])). The estimate of r_p is derived from the Fermi-gas model of the C^{13} cluster.

However, representing such a light nucleus as C^{13} as a Fermi gas is much too crude in the calculation of the factor r_p occurring in eqs. (7) and (8). The primary difficulty is that the level density of C^{13} is greatly overestimated in treating it as a Fermi gas. One can see this by comparing the calculated level density of C^{12} at an excitation energy of about 20 MeV on the Fermi gas assumption to the actual level density. (We refer to C^{12} instead of C^{13} because very few energy levels are known for C^{13} in this energy region.) The calculated [11]) level density is 250 per MeV, while the experimental [9]) level density is only 5 per MeV. In our calculation of r_p we have therefore overestimated the density of final states into which a scattering can occur by a factor of about 50 †. The final

† The calculated large values of r_p on the Fermi gas model becomes of course progressively more reliable with increasing A, because the Fermi-gas assumption is more accurate for heavy than for light nuclei. This is due to the fact that in heavy nuclei energetically favoured substructures are not as strongly expressed as in light nuclei. Thus, optical giant resonances are less likely to occur in heavy nuclei than in light nuclei. Possible resonances are further damped out in heavy nuclei because as the nuclear volume increases, the value of r_0 in eq. (6) tends to increase.

estimates are therefore

$$\tau \approx 2\times 10^{-20}\ \text{sec}, \tag{7'}$$

$$\Delta E \approx 100\ \text{keV}. \tag{8'}$$

The lifetime (7') is much larger than the transit time of the α-particle across the C^{13} nucleus, the latter being $\tau_{tr} \approx 3\times 10^{-22}$ sec for a bombarding energy of 5—10 MeV. The total width of the optical giant resonance is therefore restricted by the transit time to be about †

$$\Delta E = \frac{\hbar}{\tau_{tr}} \approx 2\ \text{MeV}. \tag{9}$$

The energy spacing between giant resonances should be some MeV as inferred from the energy separation of the α—C^{12} cluster states in O^{16} and the α—C^{13} (gd. st.) and α—N^{13}(gd. st.) cluster states in O^{17} and F^{17} (ref. [10])).

As yet, α-C^{13} elastic cross sections have not been measured, but other experimental data are discussed below (sect. 5.1), from which we are able to infer that the expectations for optical resonance phenomena stated above are substantially correct.

3.2. SCATTERING OF C^{12} ON C^{12} AND O^{16} ON O^{16}

Recently, elastic and reaction cross sections for the scattering of C^{12} on C^{12} and O^{16} on O^{16} have been measured by Almqvist, Bromley and Kuehner [12]). The remarkable feature of their results is that an optical resonance structure (a kind of giant resonance effect) is observed in the C^{12}-C^{12} scattering but not in the O^{16}-O^{16} scattering. It is of considerable interest, therefore, to attempt an explanation of these findings from our point of view.

Calculation of the lifetime of two unexcited, bound C^{12} clusters from eq. (6) gives the value ††

$$(2—5)\times 10^{-21}\ \text{sec}. \tag{10}$$

The corresponding partial energy width is of the order of 100 keV. In obtaining these values, we have reduced the value of r_p calculated from the Fermi gas model by the factor $\frac{1}{50}$, in the same way as discussed in sect. 3.1. The factor r_0 is quite small: $r_0 = 0.02$. It is determined from the spacings within the rotational band of levels in Mg^{24} occurring at 11.751 MeV (0—), 11.985 MeV (2+) and 12.531 MeV (4+) (ref. [13])). By assuming that this rotational band arised from the spectral energies of a rotator formed by two C^{12} (gd. st.) clusters, the average distance through which the C^{12} clusters move in a bound state is

† The height of the Coulomb barrier is about 3 MeV for α-C^{13} scattering. This is sufficiently smaller than the bombarding energies considered to take the penetration factor for the α-particles at the C^{13} surface to be unity.

†† By using the nucleon-C^{12} total cross section in eq. (6), an approximate account is taken of the possibility that the C^{12} clusters are internally excited.

directly calculable and leads to the stated value of r_0 †. That the rotational band from two C^{12} clusters should occur at such high energies ($\approx$12 MeV) in the spectrum of Mg^{24} is due to the Pauli principle. We can see this as follows. The formation of C^{12} cluster state requires strong correlations among the single particle states in Mg^{24}. At low excitation energies, these states will not be available (because of the Pauli principle). At higher excitation energies, however, there will be a sufficient number of single particle states available to form the C^{12} clusters, and because of their rather large binding energy, the formation of these cluster states will be energetically favoured. The large binding energy also inhibits internal excitations so that the C^{12} clusters can be taken in their ground states. Note that the separation energy of the bound C^{12} clusters in the lowest rotator state (i.e. the 11.751 MeV state of Mg^{24}) into two free C^{12} clusters is, therefore, only about 9 MeV below the top of the Coulomb barrier for C^{12}-C^{12} scattering (6.6 MeV); we utilize this result in our discussion of O^{16}-O^{16} scattering below.

The expectations for C^{12} scattering on C^{12} may now be stated. For incident energies well below the Coulomb barrier all resonance effects are damped out. The elastic cross section will be pure Mott scattering and the reaction cross sections will be very small. At energies just below and above the Coulomb barrier, optical resonance effects should appear in the scattering and therefore in the reaction cross sections ††.

In the region of the Coulomb barrier an entrapped cluster undergoes many reflections in the nucleus so that the widths of the resonances should be about 100 keV as determined from eq. (10). The energy interval between resonances should be about 250—500 keV, the same separations as in the aforementioned rotational band. As the energy of the incident C^{12} particles is increased further, the resonance structure should smear out and eventually disappear. This comes about because of the decreasing transit time. At high scattering energies the C^{12} nuclei spend little time near each other ($\lesssim 5\times 10^{-23}$ sec) as the reflection coefficient at the nuclear surface approaches zero. The description presented is in qualitative agreement with the experimentally measured cross sections, especially with the reaction cross-sections [12]).

The fact that no resonance effects occur in the scattering of O^{16} on O^{16}, as mentioned, must be attributed to the smallness of the effective interaction between two O^{16} nuclei compared to that for two C^{12} nuclei. The situation can be understood as follows. By the very structure of the clusters, two O^{16} clusters cannot overlap as well as two C^{12} clusters. This is again due to the Pauli

† That the two C^{12} clusters form a rotator does not mean that these clusters are at a fixed separation distance. On the contrary, the clusters will oscillate very rapidly with an average kinetic (and potential) energy of about 50-100 MeV ref. [7])).

†† Since the reaction cross sections depend directly on the capture probability of the incident particle, any resonance behaviour associated with the incoming channel of a nuclear reaction should be usually exhibited in all allowed reaction channels.

principle. Nucleons in the C^{12} clusters can penetrate into their mutual p-states while this can occur in O^{16} only by intermally exciting (i.e. breaking up) the O^{16} cluster. This small overlapping results in a relatively small effective interaction between two proximate O^{16} clusters and favours a large relative kinetic energy. Further, since we have already noted that the separation energy of the 11.751 M3V state of Mg^{24} into two free C^{12} nuclei is only 2.23 MeV, it is not surprising that two unexcited O^{16} clusters do not form a bound or quasi bound state below their Coulomb barrier †. This being the case, no optical resonance phenomenon is expected in the scattering of O^{16} on O^{16}. That no broad resonances are observed above the Coulomb barrier probably indicates that the effective interaction potential is very smooth, lacking a refractive edge.

Our explanation for optical resonance phenomena occurring in C^{12}-C^{12} reactions and not in O^{16}-O^{16} reactions is very similar to the explanations already proposed by Vogt and McManus [14]) and by Davis [15]) in that the effective interaction between C^{12} nuclei is larger than that between O^{16} nuclei. Our viewpoint is different in an essential way, however, with important consequences.

In the considerations of Vogt and McManus, the larger effective interaction energy in the C^{12}-C^{12} system compared to that in the O^{16}-O^{16} system is attributed to the larger rigidity of the O^{16} cores. Thus when two C^{12} clusters approach each other, the formation of loosely bound quasi-molecular states is possible; for O^{16} this does not occur. Spacings and widths of the observed resonances in the C^{12}-C^{12} elastic cross section are then accounted for by making a reasonable choice for the attractive interaction betwen these nuclei.

In the present considerations, we repeat, the Pauli principle plays an important role ††. When two nuclear clusters come together, the clusters immediately engage in a fast exchange of nucleons (exchange times of order 10^{-23} sec as given by eq. (4)). Even a large relative orbital anular momentum does not prevent this exchange, contrary to the assumption made by Davis [15]). The orbital angular momentum primarily affects the widths of the optical giant resonances through the overlap factor r_0 in eq. (6), since this factor decreases along with the interpenetation of the clusters with increasing orbital angular momentum. The Pauli principle enters through the required anti-symmetrization of the cluster wave function (5), and is represented in eq. (6) by the introduction of the factor r_p. The anti-symmetrization process accounts for the rapid nucleon exchange between the nuclear clusters and is also a determining factor in describing the coalescence of two nuclear clusters into a compound nuclear state [24]) (without destroying the clusters).

† This argument is similar to that accounting for the circumstance that Be^7 is bound while Be^8 is not [7]).

†† Of course it is the Pauli principle that is also responsible for the O^{16} clusters being more rigid than the C^{12} clusters.

It is also apparent from the nature of our considerations that optical giant resonance phenomenona and optical interference phenomena should be a very general feature of nuclear reactions and should occur in many heavy ion reactions. One should expect optical resonance phenomena, for example, in the scattering of α-particles, tritons, He^3 and even deuterons from O^{16}. Giant resonances occurring near the top of the Coulomb barrier should have a relatively small width (about 100 keV), as approximately determined by eq. (6), except perhaps for deuteron scattering. At higher scattering energies the widths of the resonances should become very broad (in the MeV range). It is always necessary, as mentioned before, that if the latter broad giant resonances are to occur, the slope of the effective interaction between the nuclear clusters must not be too smooth.

4. Optical Resonance Phenomena in Outgoing Reaction Channels

Our previous considerations have dealt with optical giant resonance phenomena only when the resonances are associated directly with the incoming channel of the nuclear reaction. Under certain circumstances, however, one should expect optical resonance phenomena to occur in a specific outgoing channel in a nuclear reaction even if resonances do not occur in the capture cross section†. This comes about as follows. Consider a nuclear reaction which is initiated, for example, by the capture of a nucleon with an energy of several MeV by a medium heavy or heavy nucleus. After the capture of the bombarding particle, the highly excited compound nucleus formed goes through all the modes of excitation allowed by conservation laws [16,17] ††. We shall refer to these different excitation modes as resonating group states [17] or simply cluster states [7]. For example, suppose that the compound nucleus allows the formation of two excited or unexcited cluster states with a finite probability; these states are described by wave functions of the general form

$$\psi = \mathrm{A}\{\varphi_a(\mathbf{r}_1 \ldots \mathbf{r}_k)\varphi_b(\mathbf{r}_{k+1} \ldots \mathbf{r}_n)X(\mathbf{R}_a - \mathbf{R}_b)\}, \tag{11}$$

where the notation is the same as in eq. (5) †††. If the lifetime of a cluster state as (11) is of the order of 10^{-21}—10^{-22} sec, or larger, and if the energy for the compound nucleus decaying into the corresponding free nuclei (excited or unexcited) is about equal to or greater than, the Coulomb barrier, conditions

† That this should be the case was first pointed out by R. Sherr (private communication).

†† The assumption of a bombarding energy of several MeV allows the nucleon to enter the nucleus through many incoming channels, especially if the energy spread of the beam is not too small. It also means that many outgoing channels are open for the subsequent decay of the compound nucleus.

††† The immediate (not successive) decay of a compound state into three nuclei is rare compared to the decay into two nuclei [18, 19, 7].

are favourable for a giant resonance structure to occur in the energy dependence of the appropriate partial decay cross section.

A true optical resonance behaviour in the reaction channels can easily be obscured or mistakenly identified. The relevant cross section is always the ratio of the partial reaction cross section to the total reaction cross section, as a function of energy. Otherwise the optical effects we are concerned with will be superposed with variation in the capture probability of the bombarding particle with incident bombarding energy.

An important qualification for the existence of true optical resonance phenomena in an outgoing reaction channel is that the probability for the highly excited compound nucleus decaying into particular decay modes does not depend sensitively on the energy of the bombarding particle (which is about 10 MeV or so). An optical resonance in a reaction cross section exhibits the energy dependence in the transmission coefficients of the observed nuclear clusters through the Coulomb barrier of the compound nucleus composed of the bombarding and target nuclei †. If the probability for formation of the pair of clusters in a particular reaction channel were to change markedly as a function of the incident bombarding energy, all resonances in the reaction cross sections would be a complicated superposition of this energy dependence in the formation probability with optical effects. This difficulty should not usually arise. Since the effective interaction energy between two clusters is usually much larger than the bombarding energy of the captured particle (say, 50 MeV and more †† compared to about 10 MeV) the probability for forming a pair of nuclear clusters should be essentially constant over the bombarding energy range considered. Thus, we can expect that, at least to a first approximation, the structure of giant resonances in partial reaction cross sections should be independent of the manner of formation of the compound nucleus. Furthermore, these giant resonances should be very similar to the giant resonances occurring in the appropriate reverse elastic scattering processes †††, excluding Coulomb effects.

It is very difficult to estimate the magnitudes of the partial reaction cross sections, because this would require knowing the reaction mechanism for the creation of the different cluster structures allowed. That is, one would have to know the time development of the compound nuclear system starting from the capture of the bombarding particle. One can only expect that these partial cross sections are much smaller than the appropriate total cross section

† We refer only to resonances in the reaction channels not associated with the capture cross section.

†† For example, as remarked in sect. 3.2., the interaction energy of two C^{12} clusters is about 50-100 MeV.

††† In this comparison one should consider the ratio of the partial to total reaction cross sections.

(say by a factor of $\frac{1}{20}$ or less) and much larger than what any statistical model predicts †.

5. Discussion of Examples

5.1. THE O^{16}(gd. st.) (p, α) N^{13}(gd. st.) REACTION

This reaction was investigated for proton energies from 6 MeV to 20 MeV by Whitehead and Foster [20]) and by R. Sherr and his collaborators [21]). It is observed that the partial cross section for this reacrion displays giant resonances with widths of 1—2 MeV. (Also, the O^{16} (p, p) O^{16} cross section has a smooth behaviour in this energy range [21]).) The peak of the first giant resonance occurs for a proton energy of about 8.5 MeV corresponding to a decay energy of 3 MeV. The energy separations between resonances are a few MeV.

As discussed in sect. 4, the same optical resonance structure should appear in the appropriate reverse scattering process and its mirror reaction, i.e., the elastic scatterring of α-particles on N^{13} and C^{13}, respectively ††. The latter process was discussed in subsect. 3.1 and the conclusions arrived at there are in accord with the giant resonances observed by White read and Foster and by Sherr.

It would be very interesting to investigate experimentally to what extent the structure of the giant resonances associated with the outgoing channels of a nuclear reaction is ıeally independent of the way in which the compound nucleus system is formed. To this end, one could investigate such reactions as

$$N^{15}(d, \alpha)C^{13}(\text{gd. st.}), \qquad N^{14}(t, \alpha)C^{13}(\text{gd. st.}), \qquad \text{etc.}$$

5.2. THE Al^{27} (n, α) Na^{24} REACTION

This reaction has been investigated by H. W. Schmitt and J. Halperin [22]) for neutron energies ††† from 6.1 MeV to 8.3 MeV. An optical giant resonance structure is observed with widths roughly of about 100 keV and energy separations between resonances of about 0.5 MeV. These values are about the same a those found in the giant resoance spectrum in the C^{12}-C^{12} scattering discussed in subsect. 3.2. especially with regard to the C^{12} reaction cross sections. This correspondence is expected when one notes that the Coulomb barrier for the α-Na^{24} scattering is about 4—5 MeV. Thus the energy into the α-Na^{24} decay channel in the above experiment extends from a little below to just above the Coulomb barrier, This indicates that it is the lifetime of the α-particle cluster in the Na^{24} nucleus, and not its transit time, that determines the widths of the giant resonances. In all our previous estimates the lifetime of such an α-particle

† This is also in part due to the fact that the indistinguishability of the nucleons greatly reduces the differences between different cluster wave functions.

†† However, N^{13} is not stable.

††† The decay energies range from 3 MeV to 5 MeV, providing Na^{24} is not excited after the decay [22]).

cluster in a light nucleus was found to be about $(2—5) \times 10^{-21}$ sec, corresponding to an energy width of some 100 keV.

The experimentally observed giant resonances would probably be best exhibited by measuring decays to a specific state of the residual nucleus as, for example, the ground state of Na^{24}.

6. Summary

We have discussed why one should expect optical giant resonance phenomena to occur in the cross sections for the scattering of composite particles from nuclei, a circumstance very analogous to the well known optical resonance phenomena occurring in the scattering of nucleons from nuclei. For the existence of giant resonances in composite particle scattering it is most essential that the appropriate cluster structures live long enough in a compound nuclear state. That this is often the case is a result of the indistinguishability of the nucleons in the compound nuclear state, which prevents the rapid destruction of the cluster structures. Further, using the same reasoning we conclude that certain outgoing reaction channels should similarly exhibit these optical resonance effects † even though the capture cross sections do not. Giant resonances should even occur in the decay of a compound nucleus to certain excited decay products. At scattering or decay energies considerably above the Coulomb barrier, an optical resonance effect can only occur if the slope of the effective interaction between the nuclear clusters is not too smooth.

In conclusion, it should be mentioned that our considerations are made completely in the spirit of the compound nucleus picture of Niels Bohr [16]), especially if one considers the formulation by Wheeler of the resonating group theory [17]).

† The explanation of this effect is very analogous to the explanation of asymmetric fission [23]).

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THE NUCLEUS Y^{88} AND THE $Y^{88}-Sr^{88}$ MASS DIFFERENCE

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Abstract: The Y^{88}–Sr^{88} mass difference has been determined by measuring threshold energies for Sr^{88}(p, n)Y^{88} (4454±6) keV and Sr^{88}(p, n)Y^{88*} (γ) (4848±6) keV. The latter value when combined with our determination of the Y^{88*} (γ) energy (393±1) keV leads to a ground state threshold energy of 4450±6 keV. These measurements result in an Y^{88}–Sr^{88} mass difference of 3619±6 keV. In order to attempt to understand a 200 keV discrepancy between the (p, n) threshold energy and the $Y^{88}(\beta^+)$ Sr^{88*} end-point energy, searches were made for a (p, n) threshold at bombarding energies of 4.1 to 4.45 MeV, but none was found. A search was also made for a γ-ray in the neighbourhood of 200 keV. No evidence for new states in Y^{88} or Sr^{88} was found. Comparison with experiment of theoretical capture-to-positon ratios for the transition from Y^{88} to the 1837-keV state of Sr^{88} based on the above mass difference and a study of log *ft* values for this transition favour spin and parity assignments of 4^- and 1^+ for the Y^{88} ground and first excited states, respectively.

1. Introduction

For some time now there has existed a puzzle concerning the weak (0.2 %) positon decay of Y^{88} to the first excited level of Sr^{88} at 1.84 MeV (ref. [1]). Various groups have studied the Y^{88} positon spectrum using different techniques and have arrived at different results [2–5]). This caused an uncertainty in the Y^{88}–Sr^{88} mass difference since prior to (p, n) threshold measurements the positon decay studies were the most accurate means of establishing this mass difference.

The two main problems connected with the positon measurements are to determine the end-point energy accurately and to determine the shape of the positon spectrum. Peacock and Jones [2]) at Oak Ridge were the first to have studied Y^{88} positon decay. They used a thin lens beta-ray spectrometer and obtained an allowed shape with an end-point energy of 830 keV. Later Stirling [3]) pointed out that the spin of the Y^{88} ground state should be 4^- or 5^- according to shell theory and that the Nordheim strong coupling rule gives 4^-. (The configurations should be $g_{\frac{9}{2}}^{-1}$ for the neutrons and $p_{\frac{1}{2}}$ for the protons. These configurations are found in the neighbouring nuclei ${}_{38}Sr_{49}^{87}$ with a $(g_{\frac{9}{2}}^{-1})$ neutron configuration and ${}_{39}Y_{50}^{89}$ with a $(p_{\frac{1}{2}})$ proton configuration. The 4^- choice is also given by the Brennan-Bernstein [6]) coupling rule R1, which is right in 22 out of 22 cases assuming it is right for Y^{88}. If the Y^{88} ground state were 4^-,

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the positon spectrum should have the unique first forbidden shape instead of the allowed shape reported by Peacock and Jones [2]), so Stirling set out to remeasure the positon spectrum. He also found an allowed shape and positon end-point energy of 830 keV in excellent agreement with Peacock and Jones. However, he and Goldberg [3]) remarked that 0.83 MeV $\approx$ 1.84—1.02 MeV, which is the maximum kinetic energy for a positon arising from internal pair production due to de-excitation of the 1.84-MeV state in Sr^{88}. They concluded that the positons from Y^{88} decay were being masked by positons from internal pair production. After making a rather uncertain correction for this effect they concluded that their results were consistent with a unique first forbidden spectrum with an end-point energy of 600 keV. They remarked that if they had known what the end-point energy should have been, the internal pair correction would have been much more straightforward.

Ramaswamy and Jastram [4]) then decided to measure the positon spectrum by a different method. They used a 4π scintillation technique. The Y^{88} source was sandwiched in between thin anthracene crystals which were viewed by an EMI 9536B photomultiplier. Y^{88} positons were detected in coincidence with one annihilation γ-ray and a 1.84 MeV γ-ray, thus eliminating the effect of internal pairs. They obtained the unique first forbidden shape as expected but the end-point energy was measured to be 560 ± 30 keV. They also measured the L_1/K ratio for Y^{88} capture to the 2.74 MeV Sr^{88} state and as a result of this measurement deduced a decay energy of 540 ± 160 keV. A positon end-point energy of 420 ± 160 keV was thus implied.

Next, a group [7]) at Florida State University, using a tandem Van de Graaff accelerator, reported the Sr^{88}(p, n)Y^{88} ground state threshold to be 4468 ± 14 keV. This leads to an end-point energy of 776 ± 14 keV in marked disagreement with Ramaswamy and Jastram [4]). Meanwhile, Rhode *et al.* [5]) at Purdue using a 4π scintillation technique have done an experiment very similar to Ramaswamy's but they have measured the positon spectrum using photomultipliers and a summing circuit in coincidence with two annihilation γ-rays as well as the 1.84 MeV γ-ray. They obtained an allowed shape and not a unique first forbidden shape. Therefore they favour 3^- for the Y^{88} ground state spin. However, their end-point energy of 766 ± 10 keV agrees well with the (p, n) threshold measurements.

In order to settle certain points in this puzzle we have remeasured the Sr^{88} (p, n) ground state threshold energy and have searched in the region below it for a possible threshold [8]) which would correspond to the positon end-point energies measured by Ramaswamy and Jastram and Stirling and Goldberg. We have studied the γ-ray spectrum arising from the proton bombardment of strontium in the neighbourhood of the Y^{88} ground state threshold energy and the Y^{88} first excited state threshold energy, to see if any unaccounted for low energy γ-rays might occur. Finally, we have obtained the threshold for the

first excited state of Y^{88} by observing the yield of the 393-keV Y^{88} γ-ray as a function of incident proton energy. This determination coupled with a careful measurement of the γ-ray energy leads to an independent Y^{88}–Sr^{88} mass difference determination.

2. Experiment

2.1. GROUND STATE (p, n) THRESHOLD DETERMINATION AND SEARCH FOR A PREVIOUSLY UNOBSERVED LOWER ENERGY THRESHOLD

The Bartol-ONR Van de Graaff Generator supplied a beam of protons of 0.1—1μA for the determination of the Sr^{88}(p, n)Y^{88} threshold energy. Energy calibration was done using the C^{13}(p, n) N^{13} and F^{19}(p, n) Ne^{19} reactions. Thick carbon targets were prepared by drying an aquadag solution with a heat lamp on a thick gold disk. Fluorine targets were prepared by dripping hydrofluoric acid on a thick aluminium disk. The strontium targets were prepared by evaporating metallic strontium as well as SrO_2 powder onto thick gold. The targets were placed at the end of an aluminium cylinder 1.27 cm in diameter and 3.81 cm long and were held in place by a tantalum liner.

The slow-fast counter ratio method was used to determine the (p, n) thresholds [9]). The slow counter consisted of a counter tube 2.54 cm in diameter filled with enriched $B^{10}F_3$ and surrounded by a layer of paraffin 5 cm long and 1.3 cm thick. It was set with its axis vertical and at a distance of 3.7 cm from the target. The fast counter was essentially a conventional long counter. It was placed with its front face 27.4 cm from the axis of the slow counter. Using the value for the C^{13}(p, n) threshold of 3235.7$\pm$1.2 keV (ref. [10]), we obtained for the F^{19}(p, n) threshold 4235+5 keV as compared to the accepted value [10–11]) 4233$\pm$2. The ground state threshold for Sr^{88}(p, n) was determined to be 4454$\pm$6 keV from series of four measurements where the F^{19}(p, n) threshold was determined just before the Sr^{88}(p, n) threshold so that effectively the Sr^{88}(p, n)–F^{19}(p, n) threshold difference was determined. This was measured as 221$\pm$6 keV which combined with the accepted value for the F^{19}(p, n) threshold of 4233$\pm$2 keV gives 4454$\pm$6 keV for the Sr^{88}(p, n) threshold. The Sr^{88}(p, n) threshold was also measured relative to the C^{13}(p, n) threshold and a value of 4455$\pm$6 keV was obtained. Thus our best value for the Sr^{88}(p, n)Y^{88} ground state threshold from these measurements is 4454$\pm$6 keV. This result is entered in table 1. All measurements were corrected for a proton beam energy spread of 6 keV. No target thickness corrections were applied, but the carbon targets were known to be thick, and the strontium and fluorine targets were of undetermined thickness. However, the resultant uncertainties in the thresholds should be less than 2 keV. Fig. 1 shows a Sr^{88}(p, n)Y^{88} slow-fast ratio curve. The Y^{88} ground state thresholds occurs at 4456 keV. If only the slow counter yield is plotted, it occurs at 4455 keV. Measurements were taken down to 350 keV

below threshold and no evidence for a previously unobserved threshold is obtained. The slight bump at 4.32 MeV is not statistically significant and did not repeat in other runs.

2.2 FIRST EXCITED STATE (p, n)γ THRESHOLD AND SEARCH FOR LOW ENERGY γ-RAYS

The threshold for production of the 393-keV γ-ray was obtained by extrapolating its yield to zero. It was found to occur at $E_p = 4848 \pm 6$ keV. The magnet was calibrated against the C^{13}(p, n) thresholds for this work. The target was strontium metal evaporated onto a thin gold layer backed by formvar. The target thickness is estimated to be less than 1 keV. The proton beam

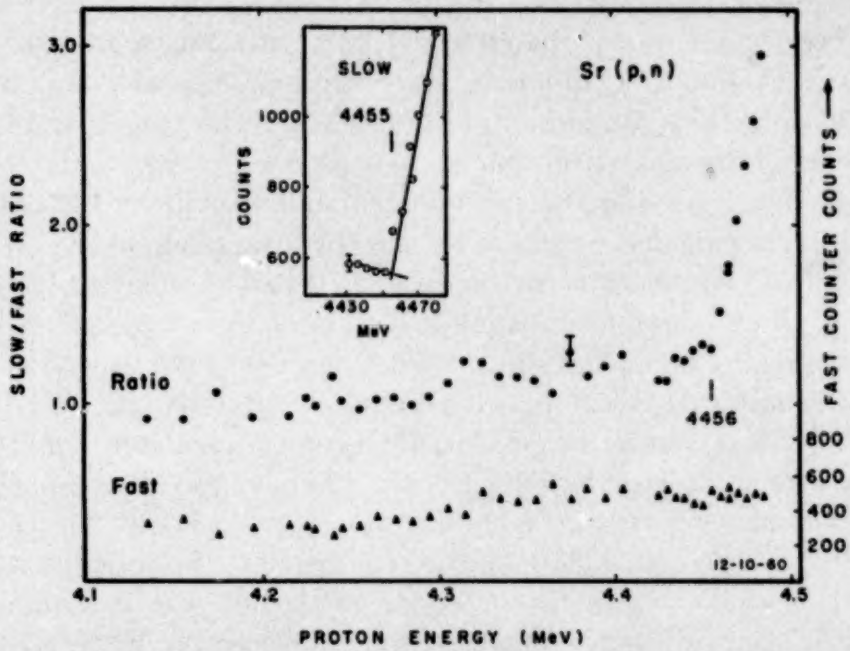

Fig. 1. Slow-fast ratio versus E_p (proton bombarding energy in the laboratory system). Inset shows slow counter yield versus E_p. This figure exhibits the results of a typical run, and shows that no addtional neutron threshold other than the one at $E_p = 4.456$ MeV is observed between $E_p = 4.1$—4.45 MeV.

passed through the target and was stopped 43 cm behind it by a tantalum disk †. A scintillation counter consisting of a 35 mm diameter by 40 mm long NaI(Tl) crystal optically coupled to a Dumont 6292 photomultiplier tube was used to detect the γ-rays. The pulse-height spectrum after amplification was observed with a RIDL 100-channel analyzer. Since the energy for the first excited to ground state transition for Y^{88} as determined from conversion electron measurements was subject to some doubt [13]), a careful determination of the γ-ray energy was made and yielded $E_\gamma = 393.2 \pm 0.8$ keV. The sources

† This arrangement is described in an article by D. M. Van Patter, R. Rikmenspoel and P. N. Trehan [12]).

used as standards were Hg^{203} ($E_\gamma = 279.12 \pm 0.5$ keV) and Cs^{137} ($E_\gamma = 661.65 \pm 0.10$ keV). The yttrium γ-ray was produced while the calibration sources were also irradiating the NaI(Tl) crystal in order to avoid counting rate effects. The Y^{88*} γ-ray energy is in good agreement with later conversion electron measurements [14]). This γ-ray energy combined with the Y^{88} excited state threshold energy leads to a ground state threshold energy of 4450 ± 6 keV. This energy averaged with the directly determined energy of the ground state

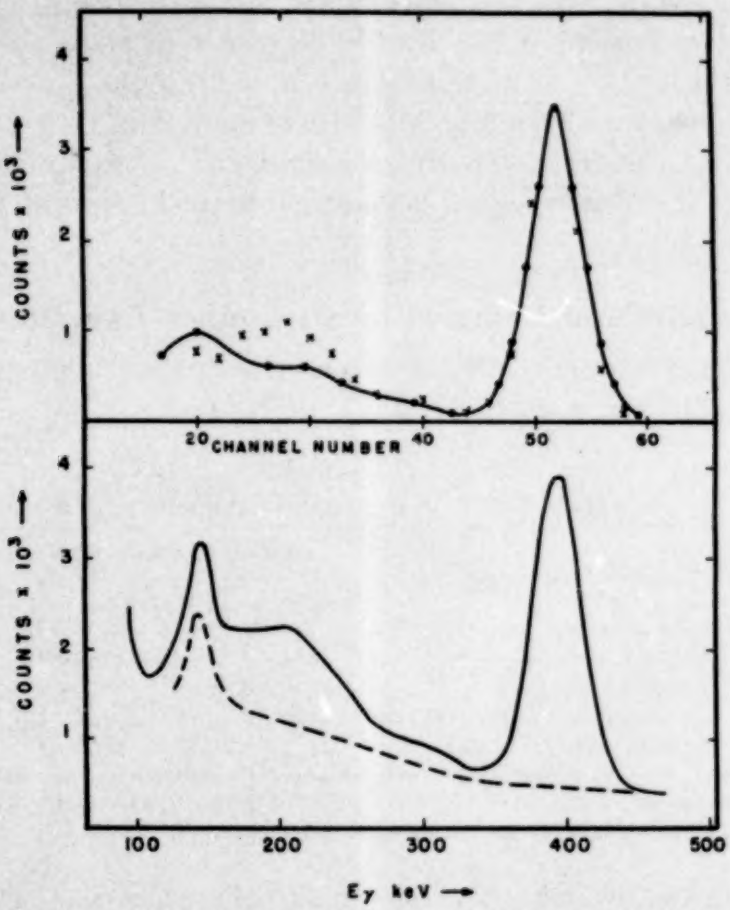

Fig. 2. Bottom: Gamma-ray spectrum obtained when a thin SrO_2 target on a gold reinforced formvar backing is bombarded by 5-MeV protons. The lowest curve is the estimated background due to bremsstrahlung, back-scattering and Compton-tails from higher energy γ-rays, neutrons, γ-rays arising from proton bombardment of the target backing etc. Top: Expected shape of a single 393-keV γ-ray (solid curve). The X indicate differences between the observed γ-ray spectrum and the estimated background. After background subtraction, the γ-ray intensity in the 200 keV region is <10 % of the 393-keV γ-ray intensity.

threshold gives 4452 ± 6 keV as the best value for the $Sr^{88}(p, n)Y^{88}$ threshold energy from this work.

Even though the γ-ray spectrum from Zr^{88} which decays to the 0.3 msec state of Y^{88} has been observed [13]) it was felt worthwhile to study the spectrum below 500 keV arising from the $Sr^{88}(p, n)Y^{88*}(\gamma)Y^{88}$ reaction using the methods of γ-ray analysis in order to place a limit on the intensity of low energy γ-rays

present. Thus various attempts were made to observe a γ-ray of energy $\approx$ 200 keV in order to try to understand Ramaswamy and Jastram's low positon end-point energy. Fig. 2 shows the results of one such attempt at $E_p \approx 5.0$ MeV. It also shows the estimated background curve. This arises due to bremsstrahlung, back-scattering and Compton-tails from higher energy γ-rays, neutrons, γ-rays arising from proton bombardment of the target backing, etc. Fig. 2 (top) shows the expected shape of a pure 393-keV γ-ray as well as the difference between the estimated background and the experimentally observed pulse-height distribution. It can readily be seen that these two follow each other quite closely. In fact after background subtraction no γ-ray of energy greater than 100 keV which has an intensity of more than 10 % of the 393-keV γ-ray intensity is observed. Spectra of a similar nature were obtained at lower proton energies in order to see if a 200-keV γ-ray might appear. No such γ-ray was observed.

3. Results and Comparison with other Experiments

Table 1 summarizes the results of the present work and shows that they are in agreement with the work of Nelson *et al.* [7]).

TABLE 1
Laboratory (p, n) thresholds for Sr^{88}

Threshold measured	Threshold energy (keV)	
	Present work	Nelson *et al.* [7])
Ground state	4454±6	4468±14
Excited state	4848±6	4852±10
Ground state (derived) [a])	4450±6	4454±10
Ground state (average)	4452±6	4460±10

[a]) The value of the excited state threshold was combined with the measured value of the γ-ray energy (393±1 keV) to obtain "ground state (derived)" threshold values.

Table 2 gives the available Y^{88}–Sr^{88} mass difference data. This difference is obtained from the laboratory Sr^{88}(p, n) Y^{88} threshold energy by transforming

TABLE 2
Y^{88}–Sr^{88} mass difference (atomic mass scale)

Reaction		Particle or radiation detected	Ref.	ΔM (keV)
Sr^{88}(p, n)Y^{88}	Sr^{88}(p, n')Y^{88*}(γ)	n γ	[a])	3619± 6
Sr^{88}(p, n)Y^{88}	Sr^{88}(p, n')Y^{88*}	n n'	[7])	3627± 10
Y^{88}(β^+)Sr^{88*}(γ)		β^+, γ coinc.	[5])	3625± 10
Y^{88}(β^+)Sr^{88*}(γ)		β^+	[2])	3690± 20
Y^{88}(β^+)Sr^{88*}(γ)		β^+	[3])	3440±100 [b])
Y^{88}(β^-)Sr^{88*}(γ)		β^+, γ coinc.	[4])	3420± 30

[a]) Present work. [b]) Error estimated by author since none is quoted.

to the centre-of-mass system and subtracting the neutron-hydrogen atom mass difference 782.6±0.4 keV (ref. [15]). It is obtained from the Y^{88} positon decay end-point energy by adding $2m_0c^2$ plus the excitation energy of Sr^{88}(1837±2 keV). This value for the Sr^{88} excitation energy is obtained by taking a weighted average of recent γ-ray energy measurements [16, 17]). The Y^{88}–Sr^{88} mass differences inferred from (p, n) and (p, n γ) threshold measurements are seen to be in good agreement with the mass difference resulting from the positon end-point energy measurement of Rhode *et al.* [5]) and in sharp disagreement with the mass differences inferred from the positon end-point energy determinations of Ramaswamy and Jastram [4]) and Stirling and Goldberg [3]).

TABLE 3

Y^{89}–Sr^{88} mass difference (O^{16} scale)

Method		Ref.	$\Delta M-1$ AMU (keV)
[Zr^{90}–Sr^{88}		[1])	
Y^{89}(d, p)Y^{90}	$Y^{90}(\beta^-)Zr^{90}$]	[18, 19])	417± 55
Sr^{88}(d, p)Sr^{89}	$Sr^{89}(\beta^-)Y^{89}$	[20, 21])	380± 60
Sr^{88}(p, n)Y^{88}	$Y^{89}(\gamma, n)Y^{88}$	[a]) [22])	398± 80
$Y^{88}(\beta^+)Sr^{88*}$	$Y^{89}(\gamma, n)Y^{88}$	[2, 22])	467± 83
$Y^{88}(\beta^+)Sr^{88*}$	$Y^{89}(\gamma, n)Y^{88}$	[5, 22])	403± 80
$Y^{88}(\beta^+)Sr^{88*}$	$Y^{89}(\gamma, n)Y^{88}$	[3, 22])	217±128
$Y^{88}(\beta^+)Sr^{88*}$	$Y^{89}(\gamma, n)Y^{88}$	[4, 22])	197± 85

[a]) Present work

Table 3 gives values for the Y^{88}–Sr^{88} mass difference, calculated on the basis of different mass links. The first entry is obtained by combining the recent mass spectrometer measurement of the Zr^{90}–Y^{88} mass difference [1]) with the Zr^{90}–Y^{89} mass difference obtained from the Q value of the reaction Y^{89}(d, p) Y^{90} measured by Wall [18]) and the beta disintegration energy of $Y^{90}(\beta-)Zr^{90}$ measured by Langer and Price [19]). The second entry is obtained by combining the Q value [20]) of the Sr^{88}(d, p)Sr^{90} reaction and the beta disintegration energy [21]) of $Sr^{89}(\beta^-)Y^{89}$. The next five entries are obtained combining the Y^{89}–Y^{88} mass difference obtained from the $Y^{89}(\gamma, n)Y^{88}$ threshold measurement of Geller *et al.* [22]), with the Y^{88}–Sr^{88} mass differences of table 2. In every case the Y^{89}–Sr^{88} mass difference involves at least two independent measurements. One of these usually involves an error of the order of 100 keV. In the case of the first entry most of the error is due to the uncertainty in the (d, p) Q value. In the case of the second entry the Sr^{88}(d, p) Sr^{89} Q value (4.30±0.06 MeV) is responsible for most of the uncertainty. The next five entries suffer because of the uncertainty in the $Y^{89}(\gamma, n)Y^{88}$ threshold measurement [22]) ($E_{th} = \leqq 11.59\pm0.08$ MeV). Clearly it would be most helpful if the previously mentioned (d, p) Q values and the $Y^{89}(\gamma, n)Y^{88}$ threshold could be established with higher precision.

A few comments will nevertheless be made: (1) The Y^{89}–Sr^{88} mass difference computed on the basis of the Sr^{88}(d, p)Sr^{89} and $Sr^{89}(\beta^-)Y^{89}$ Q values agrees with the Y^{89}—Sr^{88} mass difference computed on the basis of the present Sr^{88} (p, n) threshold measurement and the $Y^{89}(\gamma, n)$ threshold measurement, assuming both are ground state thresholds. (2) The $Y^{89}(\gamma, n)Y^{88}$ threshold measurement and the Y^{88} positon end-point energy of Rhode *et al.* [5]) also lead to Y^{89}–Sr^{88} mass differences in agreement with the above. (3) The $Y^{89}(\gamma, n)Y^{88}$ threshold measurement combined with the Y^{88} positon end-point energy of Ramaswamy and Jastram [4]) lead to an Y^{89}–Sr^{88} mass difference which disagree with the above. This could be explained if the $Y^{89}(\gamma, n)Y^{88}$ threshold measurement were for an excited state of Y^{88} rather than the ground state. However, if this were the case the values of the Y^{89}–Sr^{88} mass difference involving it would all have to be lowered by the excitation energy of such a state. In order to restore agreement with the Sr^{88}(d, p)Sr^{89} Q-value measurements, it would be necessary to discount the Y^{88} positon end-point energy measurement of Rhode *et al.* [5]), and to assume that the Sr^{88}(p, n) Y^{88} threshold measurement also involved an excited state rather than a ground state. However, a number of cases of ground state thresholds which have clearly been observed and which involve large spin changes exist; e.g., Fe^{56}(p, n)Co^{56} and Ti^{48}(p, n)V^{48}. In both cases the target nucleus has spin 0^+ and the residual nucleus is thought to have spin 4^+. Therefore, the Sr^{88}(p, n)Y^{88} threshold which presumably involves a 0^+ target and a 4^- residual nucleus, should also be observable. For these reasons as well as others to follow we do not believe that the above explanation of Ramaswamy and Jastram's low positon end-point energy measurement is correct.

4. Discussion and Conclusions

4.1. REVIEW OF THE Zr^{88}–Y^{88}–Sr^{88} DECAY SCHEME

This decay chain is shown in fig. 3. The ground state of Zr^{88} is assumed to be 0^+ since it is an even nucleus. The half-life for the decay of Zr^{88} to the first excited state of Y^{88} by electron capture is 85 d and according to positon and beta decay systematics the decay energy should be between 0.1 and 1.1 MeV though it has not been studied experimentally †. Therefore the log ft value should lie between 4.7 and 6.8 and $\Delta J = 0$ or 1 and $\Delta\pi =$ yes or no. The spin of the first excited state of Y^{88} is therefore expected to be 1 or 0 with either parity being allowed. This state loses its excitation energy by emission of a 393-keV γ-ray with a 0.3 msec half-life. The transition is E3 according to the conversion coefficient measurements and the single particle Weisskopf formula. The ground state of Y^{88} is thought to be 4^- because of shell model

† In most cases references will not be made to individual experiments but all such references can be found in the Nuclear Data Sheets [17]).

considerations mentioned in the introduction. These facts lead to the 1^+ assignment of the Y^{88} excited state.

The primary decay mode of Y^{88} (94 %) is by electron capture to the 3^- state of Sr^{88}. The log *ft* value for this transition is 6.7. The transition should be allowed or first forbidden ($\Delta J = 0, 1$; $\Delta\pi =$ yes or no). Therefore the Y^{88} ground state spin should be 4, 3 or 2. The secondary mode will be discussed below.

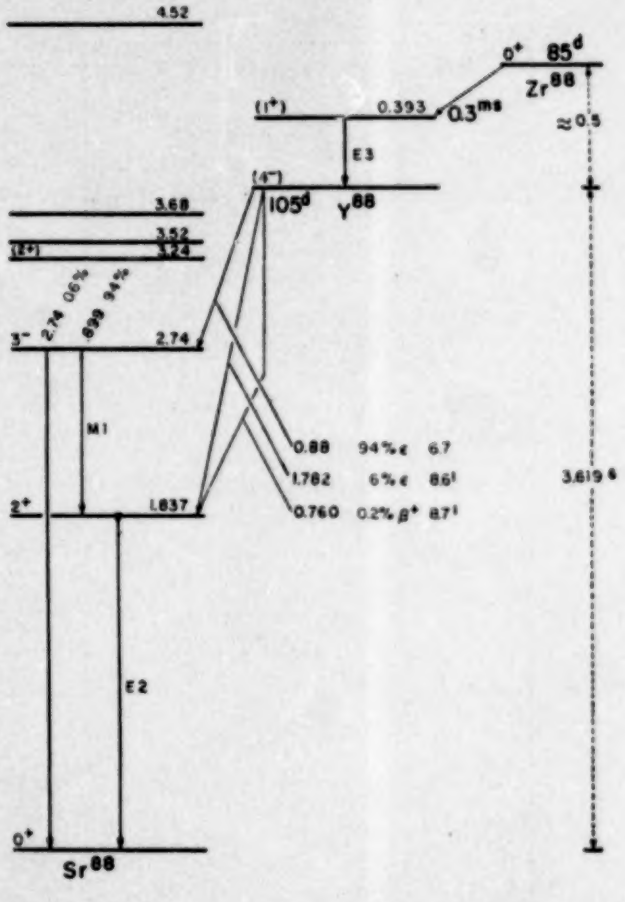

Fig. 3. Zr^{88}–Y^{88}–Sr^{88} decay scheme. The style of the Nuclear Data Sheets [17]) has generally been followed.

The spin of the ground state of Sr^{88} with 50 neutrons and 38 protons is surely 0^+ though this has never been checked directly. However, the optical spectrum of natural strontium gives no indication of hyperfine splitting attributable to electric or magnetic moments of Sr^{88}. Conversion coefficient measurements establish the 1.84-MeV γ-ray as E2 or M1 and the 0.90-MeV γ-ray as E1. Gamma-gamma angular correlation measurements are consistent with the first radiation being pure E1 and the second being pure E2. Thus the spin sequence 3^-, 2^+, 0^+ is indicated. Polarization direction correlations established this spin sequence also.

4.2. EFFECT OF RECENT γ-RAY, POSITON AND (p, n) THRESHOLD MEASUREMENTS ON SPIN AND PARITY ASSIGNMENTS FOR Y^{88}

Recently Peele [23]) has observed that the ratio of the intensities of the 1.84-MeV Sr^{88} γ-ray to the 0.90-MeV γ-ray is 1.064 ± 0.008 and the ratio of 2.74-MeV γ-ray to the 1.84-MeV γ-ray is $(5.97 \pm 0.25) \times 10^{-3}$. This information combined with Peacock and Jones' measurement of the positon to 1.84-MeV γ-ray ratio of $(1.9 \pm 0.4) \times 10^{-3}$ or Stirling's measurement of 2.8×10^{-3} leads to the specification of the secondary decay mode of Y^{88} as being 6 % electron

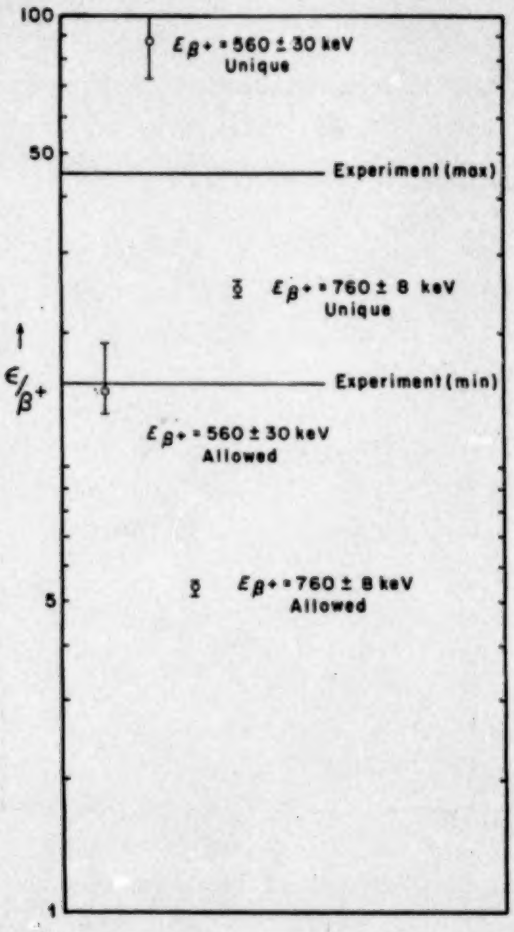

Fig. 4. Capture-to-positon ratios. The upper horizontal line is the maximum capture-to-positon ratio for the transition from Y^{88} to the 1.837-MeV state in Sr^{88} consistent with measurements of Peelle [24]) and Peacock and Jones [2]). The lower horizontal line is the minimum value for the same quantity consistent with measurements of Peelle and Stirling [3]). The vertical lines to the left are values of the ε/β^+ ratio predicted from theory assuming Ramaswamy and Jastram's [4]) end-point energy and assuming the transition is allowed (left) or unique first forbidden (next to left). The two vertical lines on the right are similar to the left-hand lines. However, an end-point energy based on the present (p, n) threshold measurement, is used.

capture and 0.2 % or 0.3 % positon emission. If the positon decay spectrum is not complex, experimental limits can be found for the capture-to-positon ratio for this Y^{88} decay mode. This ratio should lie between ≈ 45 and 16.

These limits are shown in fig. 4 by the horizontal lines. This figure also indicates the limits expected from theory [24]) for the capture-to-positon ratios based on the positon end-point energy of Ramaswamy and Jastram. If the transition were allowed or first forbidden this ratio is expected to be $\approx$ 15. If the transition is unique first forbidden the ratio is expected to be $\approx$ 88. Thus the unique first forbidden assignment is in clear disagreement with the experimental capture-to-positon ratio. If, on the other hand, an end-point energy deduced from the present work is used the capture-to-positon ratio predicted is $\approx$ 5 for an allowed or first forbidden transition and $\approx$ 25 for a unique first forbidden transition. Thus if the positon decay is not complex the unique first forbidden assignment is consistent with the experiment and the allowed or first forbidden assignment is not. However, the positon decay spectrum may be complex, according to Rhode *et al.* [5]). In this case the capture-to-positon ratio may be considerably greater than 45.

The log *ft* value for the β^+ decay of Y^{88} assuming it to be simple and to have an end-point energy of 760 keV is $\approx$ 9.4. This becomes 8.7 after correction for the unique first forbidden shape. From Y^{88} K capture one finds log *ft* $\approx$ 8.6 and the unique first forbidden correction is negligible. Slightly lower log *ft* vulues are obtained if Ramaswamy's end-point energy is used. If the decay is complex the log *ft* values will all be slightly higher but probably still consistent with the transition being unique first forbidden. Thus for all of these cases the log *ft* values are probably consistent with the transition being unique first forbidden.

If Ramaswamy and Jastram's experiment were correct with respect to the end-point energy and spectral shape, and if we wish to postulate only one new state, it would have to be either a 4^- state in Y^{88} 200 keV below the lowest state observed in the (p, n) threshold work, or a 2^+ or 6^+ state in Sr^{88} 200 keV above the 1.84 MeV state in Sr^{88}(assuming that the ground state of Y^{88} is 4^-). Since the intensity of unaccounted for 200 keV γ-radiation while the target is being bombarded is very low (fig. 2) we can rule out such a state for Y^{88}. If there were such a state the spin sequence for the three Y^{88} states would be 0 or 1, 3 or 4 and 4^- for the ground state, thus, the transition 3 or 4 to 4^- would promptly follow the 393 keV γ-ray, and would be observed. A 2^+ state in Sr^{88} at 2.04 MeV is ruled out, both because no low energy γ-rays are observed in Y^{88} decay, and because no γ-rays corresponding to transitions from the 3^- state at 2.74 MeV to a state at 2.04 MeV are observed. A 6^+ state in Sr^{88} at 2.04 MeV is also ruled out, both because the higher energy 4^- to 2^+ transition would be favoured, and because the 6^+ to 2^+ transition would be slow and positons with an end-point energy of 0.56 MeV would not be in coincidence with 1.84 MeV γ-radiation. In the case of the K/L ratio measurement of Ramaswamy and Jastram, it is worth noting that although this method of determining decay energies is very sensitive when the energy involved is only a few tens of keV,

it becomes quite insensitive for such a relatively high energy transition.

Thus it appears that a consistent description of the Y^{88} decay to the 2^+ Sr^{88} state can be made if the transition is unique first forbidden and the positon end-point energy is of the order of 760 keV. This description necessitates spin assignments of 4^- or 0^- for the Y^{88} ground state. The 0^- possibility can be ruled out because of the E3 nature of the 393 keV γ-ray. The excited state would have to be a 3^+ state and the Zr^{88} decay to it would be 3rd forbidden (log $ft \approx 18$), which is not the case. Shell model considerations also make a 0^- assignment very implausible.

It is not clear why the experiments involving studies of the positon spectrum do not agree in all respects with the above conclusions but measurements on such a weakly populated decay mode are difficult.

Since no evidence was found for new states in Y^{88} or Sr^{88} from the present investigation we conclude that the Y^{88} ground state is 3619 ± 6 keV above the Sr^{88} ground state. For the reasons stated above we assign 4^- to the Y^{88} ground state. This necessitates a 1^+ assignment for the Y^{88} first excited state.

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GENERAL EQUIVALENCE THEOREM IN VECTOR FIELD THEORY

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Abstract: A general equivalence theorem is derived for vector fields invariant under Lie groups, which shows that they are not renormalizable unless the bare mass vanishes.

Umezawa and Kamefuchi [1]) have derived an equivalence theorem for the Yang-Mills field with non-vanishing masses, and show that the theory is unrenormalizable. In this note we show that the theorem holds generally for all fields arising from local gauge transformations.

Following Yang and Mills [2]) and Utiyama [3]) we require that conservation laws are associated with gauge transformations T_i and that all interactions should be invariant under independent gauge transformations at all space-time points. It follows that the transformations T_i form a Lie group and therefore satisfy

$$[T_i, T_j] = f_i{}^k{}_j T_k, \tag{1}$$

where the $f_i{}^k{}_j$ are the structure constants of the group and obey the relations

$$\begin{aligned} f_i{}^k{}_j &= -f_j{}^k{}_i, \\ f_i{}^k{}_j f_k{}^l{}_m + f_j{}^k{}_m f_k{}^l{}_i + f_m{}^k{}_i f_k{}^l{}_j &= 0, \end{aligned} \tag{2}$$

the second following from the Jacobi identity. Using Dynkin's [4]) formulation of the theory of semi-simple Lie groups, we have proved [5]) that the structure constants also obey the relation

$$f_i{}^k{}_j = f_k{}^j{}_i = f_j{}^i{}_k. \tag{3}$$

Yang and Mills [2]) have shown that if we assume the transformation associated with a conservation law to be local in character, we are forced to introduce a non-massive vector field coupled universally to the corresponding conserved current.

Umezawa and Kamefuchi [1]) consider the Yang-Mills field which corresponds to the group of rotations in three dimensions. Taking three real vector fields $U_\mu{}^{(1)}$, $U_\mu{}^{(2)}$ and $U_\mu{}^{(3)}$ which form a vector $\mathbf{U}_\mu$ in a three-dimensional "I-space"

they write the Lagrangian for the Yang-Mills field adding a mass term

$$L = -\tfrac{1}{4}\mathbf{U}_{\mu\nu}\cdot\mathbf{U}_{\mu\nu} - \tfrac{1}{2}k^2\mathbf{U}_\mu\cdot\mathbf{U}_\mu - \tfrac{1}{2}g(\mathbf{U}_{\mu\nu}\times\mathbf{U}_\mu)\cdot\mathbf{U}_\nu - \tfrac{1}{4}g^2(\mathbf{U}_\mu\times(\mathbf{U}_\nu)\cdot(\mathbf{U}_\mu\times\mathbf{U}_\nu), \tag{4}$$

where

$$U_{\mu\nu}^{(i)} = \frac{\delta U_\nu^{(i)}}{\delta x_\mu} - \frac{\delta U_\mu^{(i)}}{\delta x_\nu}.$$

From this Lagrangian, an equivalence theorem is derived from which it is evident that the theory is unrenormalizable and that this is a result of the mass k not vanishing.

For a general Lie group of order n, and with structure constants $f_i^{k}{}_j$, we consider n real vector fields $U_\mu^{(i)}$ $(i = 1, \ldots, n)$ forming a vector $\mathbf{U}_\mu$ in an n-dimensional "I-space". Using relation (3) we find that the Lagrangian

$$L = -\tfrac{1}{2}U_{\mu\nu}^{(i)}U_{\mu\nu}^{(i)} - \tfrac{1}{2}k^2U_\mu^{(i)}U_\mu^{(i)} - \tfrac{1}{2}f_i^{k}{}_j U_{\mu\nu}^{(k)}U_\mu^{(i)}U_\nu^{(j)} - \tfrac{1}{4}f_i^{k}{}_j f_i^{m}{}_l U_\mu^{(j)}U_\nu^{(k)}U_\mu^{(l)}U_\nu^{(m)} \tag{5}$$

is invariant under the transformations of the group when $k = 0$.
Following the same method as that of ref. [1]), we derive in the normal way the interaction Hamiltonian in the interaction representation H which corresponds to this Lagrangian and find

$$H = \tfrac{1}{2}f_i^{k}{}_j U_{\mu\nu}^{(k)}U_\mu^{(i)}U_\nu^{(j)} + \tfrac{1}{4}f_i^{k}{}_j f_i^{k}{}_m U_\mu^{(i)}U_\nu^{(j)}U_\mu^{(l)}U_\nu^{(m)} + \text{normal dependent terms.} \tag{6}$$

It is well-known that the normal dependent do not contribute to observable quantities, and can therefore be dropped provided that similar terms are dropped from all internal lines in Feynman diagrams.

Writing the vector fields in the Stueckelberg formalism, we define fields $\mathbf{A}_\mu$ and $\mathbf{B}$, where

$$U_\mu^{(i)} = A_\mu^{(i)} + \partial_\mu B^{(i)}. \tag{7}$$

The commutation relations of the A_μ and B fields are

$$\begin{aligned}[A_\mu^{(i)}(x),\ A_\nu^{(j)}(x')] &= i\delta_{ij}\delta_{\mu\nu}\Delta(x-x'),\\ [B^{(i)}(x),\ B^{(j)}(x')] &= i\delta_{ij}\Delta(x-x').\end{aligned} \tag{8}$$

From Dyson's renormalization criterion the probable non-renormalizability of the theory corresponding to eq. (6) arises from the derivative of the B-field and we apply a unitary transformation to the Schrödinger equation involving this Hamiltonian with the object of removing the B-field or equivalently, replacing the $\mathbf{U}_\mu$ by the renormalizable $\mathbf{A}_\mu$.

Let us apply the transformation

$$G(\sigma) = \exp\left(-\frac{i}{k}\int_\sigma d\sigma_\mu f_j{}^i{}_k B^{(i)} U^{(j)}_{\nu\mu}(A_\nu{}^{(k)} + \gamma_\nu{}^{(k)})\right), \tag{9}$$

where $\gamma_\nu{}^{(k)}$ is a function of the operator $\alpha(B)$ and obeys the relation

$$(S(\alpha)-1)\gamma_\mu{}^{(k)}(B) = (f(\alpha)-1)\partial_\mu B^{(k)}. \tag{10}$$

The operator $\alpha(B)$ is defined by

$$\alpha(B)a^{(k)} = f_i{}^k{}_j a^{(i)} B^{(i)}, \tag{11}$$

and

$$S(\alpha) = \exp\alpha(B), \qquad f(\alpha) = \frac{\exp\alpha(B)-1}{\alpha(B)}. \tag{12}$$

The operators $S(\alpha)$ and $f(\alpha)$ have meaning only when expanded as power series in the structure constants.

After a somewhat lengthy calculation the result of this transformation applied to the above Schrödinger equation is an equivalent equation involving the Hamiltonian

$$\begin{aligned} H' = {} & \tfrac{1}{2} f_j{}^i{}_k U^{(i)}_{\mu\nu} A_\mu{}^{(j)} A_\nu{}^{(k)} + \tfrac{1}{4} f_j{}^i{}_k f_l{}^i{}_m A_\mu{}^{(j)} A_\nu{}^{(k)} A_\mu{}^{(l)} A_\nu{}^{(m)} \\ & - \tfrac{1}{2} k^2 \{2\mathbf{A}_\mu \cdot [1-f(\alpha)]\partial_\mu B + \partial_\mu B \cdot [1-f(-\alpha)f(\alpha)]\partial_\mu B\}. \end{aligned} \tag{13}$$

The first two terms of this Hamiltonian lead to renormalizable interactions, but the last term, involving k, may be seen on expansion of the operator $f(\alpha)$ to lead to unrenormalizable interactions.

That the above calculation can be performed in the general case as well as for rotations in three dimensions is a direct result of the fact that the general structure constants obey relations (2) and (3) and therefore have the same properties under exchange of indices as the tensors ε_{ijk} which are the structure constants of the group of three-dimensional rotations, and in terms of which the triple scalar products of eq. (4) may be written.

We conclude that, as a result of the non-vanishing of the mass k, all theories of this general form are unrenormalizable.

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ANGULAR DISTRIBUTION OF PHOTO-PROTONS FROM DEFORMED NUCLEI

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Abstract: The angular distribution of photo-protons emitted from deformed nuclei may be anisotropic because of several processes. The coupling of the angular momentum of the parent nucleus with those of the daughter nucleus and the emitted proton is very important for the distribution. Furthermore, the surface value of the proton wave function may vary strongly from the pole to the equator of the deformed nucleus. The proton while passing the Coulomb barrier 'feels' the anisotropy and exchanges angular momentum with the nucleus to a considerably smaller degree than an α-particle, especially with light nuclei. Angular distributions, illustrating these effects, are given for Mg^{24} and Np^{234}. Further, we approach the classical case of fixed axes, where the influence of the surface values of a proton wave function clearly exhibits itself. The total decay probabilities for the deformed nuclei considered here deviate only slightly from those of a spherical nucleus, namely by factors between 0.84 and 1.25.

1. Introduction

Several processes may give an anisotropic angular distribution of protons emitted from deformed nuclei excited by photons to energies well above the proton threshold. The penetration of the proton through the non-spherical Coulomb barrier is another of these effects. In a deformed nucleus the wave function of the proton is strongly dependent on the angles ϑ' and φ' in the coordinate system fixed to the nucleus [1]). The angular distribution of the emitted protons and also the penetration probability depend on the distribution $\varphi_0(\vartheta', \varphi')$ of the wave function on the surface of the nucleus, which lies immediately inside the Coulomb barrier. Furthermore, the angular distribution depends in a very important way on the coupling of the angular momenta I_1 of the parent nucleus, I_f of the daughter nucleus and j of the proton after the penetration.

Such anisotropic distributions have been observed. Experiments aimed at the study of these effects in the photonuclear process are in progress †.

In the following calculations we assume the photo-excitation to consist of the excitation of only one proton. We give different examples of angular distribution. Comparison of the results of such calculations for photo-excited nuclei with experiments is complex. At least in the giant resonance peak states are populated that involve linear combinations of several different single-particle excitations which means a more smeared $\varphi_0(\vartheta', \varphi')$. One may expect

† Dr. Sven A. E. Johansson, private communication

the proton emitted from an excited state in this region to show a less anisotropic angular dependence than is calculated on the basis of the assumptions mentioned. On the other hand, by calculating the angular distribution effects associated with a pure orbital as initial state, we may often be able to set an upper limit on the angular effects due to the intrinsic wave function. Furthermore, if we approximate the value $\varphi_0(\vartheta', \varphi')$ of the linear combination of several single-particle excitations by a constant value, we can calculate the anisotropy resulting from the effect of the deformed Coulomb barrier alone. This gives some account of the tendency of a linear combination of many states to give an increased proton emission along the symmetry axis. At a later stage, when more reliable information is available about a particular photo-excited state, it may be worth while to carry through the same calculations based on a more complicated proton state.

In comparing the results of the present calculation with the experimental data, one must distinguish different cases, corresponding to different regions of the proton spectrum. In its upper end, well resolved peaks can often be found, especially in the light elements. These peaks correspond to strong transitions to the ground state or the lower excited states of the daughter nucleus. In some cases these transitions might be of a rather pure single particle character, and they would then represent the type of transitions considered in the present work. They are experimentally known to show a rather strong anisotropy, and the calculations show this effect, too. The effect comes from the properties of $\varphi_0(\vartheta', \varphi')$ and from the coupling of angular momenta, while, at high energies, the effect from the non-spherical Coulomb barrier is unimportant. A comparison between the calculated and experimental angular distributions might give information about the character of these transitions.

In the low energy end of the proton spectrum the eccentricity of the Coulomb barrier grows in importance. There the spectrum is fairly continuous, being composed of a great number of single transitions. The emitted protons leave the daughter nucleus in rather highly excited states, which in general have a complicated configuration with several nucleons excited. An average over different $\varphi_0(\vartheta', \varphi')$ and over angular momenta is to be expected, and this averaging should result in an isotropic angular distribution. In contrast to this, the deformed Coulomb barrier will give a tendency in the same direction for all transitions, namely an increased proton emission along the symmetry axis, the estimate of which was mentioned above.

We compare the results for protons with Fröman's results [2]) on the alpha decay of deformed nuclei which show a strong direct influence of the deformation of the Coulomb field on the angular distribution and the half-life. Fröman assumes a constant wave function for the alpha particle on the surface of the nucleus and finds a strong tendency to penetrate at the poles of the nucleus. For protons emitted at the energies under consideration here, the influence

of the eccentricity of the Coulomb barrier is very much smaller. Especially for lighter nuclei the dominating features are the above-mentioned angular dependence of the wave functions on the surface and the coupling of the angular momenta, which are discussed in this paper for some chosen light and heavy nuclei. We find in many cases angular distributions concentrated in the equatorial plane instead of the poles.

We treat the following cases:

(1) A comparatively light deformed nucleus, Mg_{12}^{24}, an even nucleus with ground state spin equal to zero, which when excited by an E1 dipole of the photon field is characterized by $I_1 = 1$, $M_1 = \pm 1$. Three different excited states will be considered, with wave functions concentrated to the poles, the equator and the intermediate region.

(2) A heavy odd nucleus, in which the non-paired neutron and proton both possess a high spin, resulting in a high I_1. Few nuclei of this kind have been studied. According to theoretical predictions for Np_{93}^{234} the odd proton and corrresponding neutron should occupy the states $|Nn_z\Lambda\Sigma\rangle = |642+\rangle$ for the proton and $|633-\rangle$ for the neutron. The component of the angular momentum along the axis of the nucleus Ω is either 0 or 5. According to Moszkowski and Gallagher [3]), in such cases the ground state will probably correspond to the lower value of Ω. The case $\Omega = 5$ may be supposed to correspond to a long-lived isomer, which could be studied as well. For the daughter nucleus $I_f = \frac{5}{2}$. Furthermore, we make the calculations for the alignment $M_1 = I_1$, so that the axis of the nucleus is kept as much as possible along the z-axis in space, and thus the influence of the distribution of the wave function along the surface will be important.

(3) The limiting case of a nucleus with high I_1 and I_f, and small j. We treat for comparison one of the cases also treated under (1) for $I_1 = 1$. We choose the function $\varphi_0(\vartheta', \varphi')$ which has its maximum at the poles of the nucleus. For large I_1 and $M_1 = I_1$ we obtain an illustration of the influence of φ_0. Both φ_0 and the angular distribution have their maxima at $\vartheta = 0$ and π.

Inside the surface of the nucleus the proton is supposed to move in a potential of the Nilsson harmonic oscillator type. Outside the surface the potential of the nuclear forces is supposed to be zero, and only the Coulomb potential is important. In several cases the angular distribution is independent of how the joining of the potentials is made in detail.

Inside the nucleus we use the Nilsson asymptotic wave functions and determine their value $\varphi_0(\vartheta', \varphi')$ at the surface. Now the problem is to calculate the wave function $\varphi(\vartheta', \varphi')$, assuming its boundary value at the inner surface of the Coulomb barrier $= \varphi_0(\vartheta', \varphi')$, and at large distances to approach zero.

In the case of α-decay, a relation between φ and φ_0 has been given by Fröman, by Perlman and Rasmussen and by others. In the following we use the explicit formula by Fröman, which is based on a WKB-method.

When the proton penetrates the barrier and leaves the nucleus, the daughter nucleus can be left in different intrinsic states. The state of greatest probability according to this simplified picture is that of all other nucleons remaining in their states during the process. We limit ourselves to considering this case. Further, the daughter nucleus can be left in different states of collective rotation. We consider the dominant of these rotational states, that of lowest energy, the others giving only a small contribution.

2. General Theory

We use the wave functions of the unified theory. The intrinsic wave function of the daughter nucleus is called χ_{K_f}, where K_f stands for all the intrinsic quantum numbers, especially the component of the angular momentum along the axis of the nucleus: χ_{K_f} is an anti-symmetrized product of Nilsson asymptotic wave functions. The collective motion is given by $D^{I_f}_{M_f K_f}$. We use the notation

$$\phi^{I_f}_{M_f}(K_f) = \sqrt{\frac{2I_f+1}{8\pi^2}}\,\chi_{K_f} D^{I_f}_{M_f K_f}.$$

The wave function for the daughter nucleus has to be symmetrized by taking the sum for $\pm K_f$. We denote the wave function of the proton outside the nuclear surface by

$$\varphi = \sum_j f_j(r)\varphi^j,$$

where r is the distance of the proton from the centre of the daughter nucleus, and φ^j has the components

$$\varphi^j_m = [Y^1 s^{\frac{1}{2}}]_m{}^j.$$

The wave function of the complex disintegrating system is

$$\psi^{I_1}_{M_1} = \sum_j \frac{1}{r} f_j(r) \frac{1}{\sqrt{2}} \{[\phi^{I_f}(K_f)\varphi^j]^{I_1}_{M_1} + [\phi^{I_f}(-K_f)\varphi^j]^{I_1}_{M_1}\}.$$

2.1. ANGULAR DISTRIBUTION OF PHOTO-PROTONS

Before proceeding to the calculation of the different $f_j(r)$, we give the expression for the angular distribution for the case of particles with half-integer spin, which is a generalization of Fröman's calculation for spinless particles.

The probability per second per unit solid angle that a proton leaves the daughter nucleus in the state I_f, M_f, K_f in the direction ϑ is

$$P(\vartheta) = v \sum_{jj'} f_j^* f_{j'} \int [\phi^{I_f}(K_f)\varphi^j]^{I_1 *}_{M_1} [\phi^{I_f}(K_f)\varphi^{j'}]^{I_1}_{M_1}\, d\tau,$$

where we integrate over the coordinates of the daughter nucleus. In order to

treat the vector coupling of the complex conjugate states it is convenient to introduce the quantity $\bar{\psi}^j_{-m}$, defined e.g. by Bayman, [4]) as

$$\bar{\psi}^j_{-m} = (-)^{j-m}\psi_m{}^{j*}.$$

For the angular momentum coupled states we then have

$$\psi^{I_1*}_{M_1} = (-1)^{-I_1+M_1}[\bar{\phi}^{I_t}\bar{\varphi}^j]^{I_1}_{-M_1}.$$

We develop a term characterized by j and j':

$$[\bar{\phi}^{I_t}\bar{\varphi}^j]^{I_1}_{-M_1}[\phi^{I_t}\varphi^{j'}]^{I_1}_{M_1} = \sum_L (I_1I_1-M_1M_1|L0)\cdot\{[\bar{\phi}^{I_t}\bar{\varphi}^j]^{I_1}[\phi^{I_t}\varphi^{j'}]^{I_1}\}_0{}^L.$$

We may now recouple the angular momenta in such a way that $\bar{\phi}^{I_t}$ and ϕ^{I_t} are coupled together, whereby the integration over $d\tau$ becomes simple. The recoupling coefficients are denoted as the 9-j symbols. We get the series by 9-j symbols:

$$\{[\bar{\phi}^{I_t}\bar{\varphi}^j]^{I_1}[\phi^{I_t}\varphi^{j'}]^{I_1}\}_0{}^L = \sum_{p,q}((I_tj)I_1(I_tj')I_10|(I_tI_t)p(jj')q0)\cdot\{[\bar{\phi}^{I_t}\phi^{I_t}]^p\cdot[\bar{\varphi}^j\varphi^{j'}]^q\}_0{}^L.$$

Now one has

$$\int d\tau[\bar{\phi}^{I_t}\phi^{I_t}]^p = \delta_{p,0}\sqrt{2I_t+1}(-1)^{2I_t}.$$

From $p=0$ it follows that $q=L$, and further that the 9-j symbol can be reduced to

$$\frac{2I_1+1}{\sqrt{2I_t+1}}(-1)^{j'+I_1+I_t+L}\begin{pmatrix} I_t & I_1 & j' \\ L & j & I_1\end{pmatrix}.$$

Further we develop

$$[\bar{\varphi}^j\varphi^{j'}]_0{}^L = K^L_{jj'}Y_0{}^L.$$

Using the formula for the integral over the product of three Y, we find a double sum over the product of four Clebsch-Gordan coefficients, which as seen by e.g. formula 6.1.5. in Edmonds paper [5]) defines a 6-j symbol times a factor. Thus one has

$$K^L_{jj'} = (-1)^{j'-\frac{1}{2}}\sqrt{\frac{(2j+1)(2j'+1)(2l+1)(2l'+1)}{4\pi}}\begin{pmatrix} l & l' & L \\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} l & l' & L \\ j' & j & \frac{1}{2}\end{Bmatrix}.$$

As l and l' have the same parity, L must be even.

Collecting all terms and performing the reduction for $L=0$, we obtain with the notation $f_jf_{j'}/\sqrt{4\pi} = (j,j')$ the following formula for the angular distribution:

$$P(\vartheta) = v\sum_{jj'}(jj')\Big[\delta_{jj'}Y_0{}^0+(-1)^{M_1-I_t+\frac{1}{2}}\cdot(2I_1+1)\sqrt{(2j+1)(2j'+1)(2l+1)(2l'+1)}$$
$$\sum_{L\geq 2}\begin{pmatrix} I_1 & I_1 & L \\ -M_1 & M_1 & 0\end{pmatrix}\begin{Bmatrix} I_1 & I_1 & L \\ j & j' & I_t\end{Bmatrix}\begin{pmatrix} l & l' & L \\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} l & l' & L \\ j' & j & \frac{1}{2}\end{Bmatrix}\sqrt{2L+1}Y_0{}^L\Big].$$

The influence of the surface values $\varphi_0(\vartheta', \varphi')$ and of the non-spherical Coulomb field lies in the factors (jj') and the expression in the bracket is determined by the coupling of the angular momenta.

The quantum number L is restricted by the following conditions:

$$0 \leqq L \leqq 2I_1, \qquad |l-l'| \leqq L \leqq l+l', \qquad |j-j'| \leqq L \leqq j+j'.$$

We shall later calculate the angular distribution for the asymptotic case of large I_1 and I_f, while $|I_1-I_f| = a$ and j are small.

2.2. CALCULATION OF THE $f_j(r)$

The wave function describing the penetration of the proton through the Coulomb barrier has to be calculated in a coordinate system fixed in the nucleus. A transformation to the fixed system will then give $f_j(r)$. The problem is a generalization to a deformed nucleus of the usual WKB-treatment of the spherical problem, which has been performed by Christy and especially by Fröman. Fröman gives an explicit formula for the influence on the penetration of the proton by the deformation of the nucleus.

We assume that the surface of the daughter nucleus, outside of which only the Coulomb potential is important, is given by

$$R(\vartheta') = R_0[1+\beta_2 Y_{20}(\vartheta')].$$

In the case of uniform charge density the electrostatic potential is

$$V(r') = (Z-1)\varepsilon^2 \left[\frac{1}{r} + \frac{R_0^2}{r^3}\frac{3}{5}\beta_2 Y_{20}(\vartheta')\right].$$

We denote the wave function of the proton in the system fixed in the nucleus with

$$\varphi = \frac{1}{r}\sum_j \sum_\Omega g_{j\Omega}(r)\varphi_\Omega{}^j(\vartheta', \varphi').$$

On the surface, $\varphi(r, \vartheta', \varphi')$ has the prescribed boundary value $\varphi_0(\vartheta', \varphi')$. Fröman's formula for $g_{j\Omega}(r)$, when $\varphi = \varphi_0$ at the surface and zero at infinity is, adjusted to the case of the proton,

$$g_{j\Omega}(r) = R_0 \frac{G_1(E, r)}{G_1(E, R_0)} \iint \varphi_0(\vartheta', \varphi') e^{BP_2(\cos\vartheta')} \varphi_\Omega{}^{j*}(\vartheta', \varphi') d\Omega,$$

where

$$B = \beta_2 \kappa \frac{2}{\sqrt{5\pi}}\left(1-\frac{R_0 k}{2\kappa}\right)\sqrt{\frac{R_0 k}{\kappa}\left(1-\frac{R_0 k}{\kappa}\right)}, \qquad \kappa = \frac{2(Z-1)\varepsilon^2}{\hbar v}, \qquad k = \frac{p_\infty}{\hbar}.$$

We have approximately

$$\frac{G_1(E, r)}{G_1(E, R_0)} = \frac{G_0(E, r)}{G_0(E, R_0)} e^{-\frac{l(l+1)}{\kappa}\left(\sqrt{\frac{\kappa}{KR_0}}-1+\frac{1}{2}\right)}.$$

The deviation from the spherical case lies exclusively in B, so that for $B = 0$ the above expression gives the ordinary solution. We have calculated B for the two daughter nuclei Na_{11}^{23} and U_{92}^{233} considered in this paper. The energies have been chosen to correspond to the lower end of the proton spectrum. The results are given in table 1, where R_0 is given in fm, E in MeV and τ_0 in sec.

TABLE 1
Values of B

	R_0	E	E_b	τ_0	κ	B	$\frac{1}{\kappa}\sqrt{\frac{\kappa}{kR_0}-1}$
Na_{11}^{23}	3.69	1.0	4.29	3.3×10^{-20}	3.48	0.263	0.522
U_{92}^{233}	8.02	9	16.50	2.6×10^{-20}	9.70	0.444	0.094
α-decay							
Em_{86}^{222}	7.90	4.78	15.71	5.1×10^{10}	53.1	1.22	0.0477

The value of B is smaller for Em than for other nuclei in its neighbourhood because of its small eccentricity.

The fact that for the same energy κ_α is about four times larger than κ_p has great consequences: B is small for the proton and the factor $\exp[BP_2(\cos\vartheta')]$ which measures the influence of the anisotropy lies near to 1. This means that the proton while passing the Coulomb barrier does not feel the anisotropy very strongly. On the other hand this effect is not negligible and it should be possible to find it experimentally. Furthermore, the proton cannot exchange a large momentum with the nucleus, and therefore the coupling between different l is small.

Further, the lower value of κ for the proton is decisive for the relative penetration probability for different l. For Na, the quotient $|g_3(r)/g_1(r)|$ is 0.0054 and the higher quotients are still smaller. That means that if the wave function $\varphi_0(\vartheta', \varphi')$ on the surface of the nucleus contains comparable parts of Y_l and Y_{l+2}, the coefficient $g_{j\Omega}(r)$ corresponding to the latter will be unimportant outside the barrier.
For U^{233} we get

$$|g_3/g_1| = 0.392, \qquad |g_5/g_3| = 0.184, \qquad |g_7/g_5| = 0.087.$$

Thus the contribution from higher l is depressed but not negligible. In alpha decay the quotient $|g_2/g_0|$ for Em_{86}^{222} is 0.75 and $|g_4/g_2| = 0.29$, so that contributions from higher l do play a role in this case.

The transformation from the system fixed in the nucleus to the system fixed in space is given by a formula of Fröman, which is valid also for half-integer angular momenta:

$$f_j(r) = \sum_{\Omega} (-1)^{I_1-I_1+\Omega}(I_1, j, K_1+\Omega, -\Omega|I_1, K_1)g_{j\Omega}(r).$$

Because of the properties of the $\varphi_0(\vartheta', \varphi')$ considered in this paper, Ω can take on only one value.

3. Calculation of Angular Distributions and Decay Probabilities in Special Cases

3.1. THE REACTION $Mg_{12}^{24*} \to Na_{11}^{23}+p$

The nucleus Na^{23} has the deformation $\beta_2 = 0.3$—0.5. We represent the wave functions by the Nilsson asymptotic wave functions.

The protons of Mg_{12}^{24} fill all levels up to $|Nn_z\Lambda\Sigma\rangle = |211+\rangle$, $\frac{3}{2}$. When absorbing a photon e.g. in this level, it can rise to $|312+\rangle$, $\frac{5}{2}-$. This proton is the one whose penetration of the Coulomb barrier we shall study. The wave function of this excited proton defines at the surface the function $\varphi_0(\vartheta', \varphi')$, the boundary value in the solution giving the penetration. The proton left from the pair broken by the photon absorption, $|21-1-\rangle$, determines the spin and parity of the daughter nucleus, Na_{11}^{23}. This gives $I_f = \frac{3}{2}$, $K_f = -\frac{3}{2}$.

We treat this case and two analogous cases giving different φ_0, one having a maximum at the equator, one at the poles and one intermediate.

The eccentricity of the nucleus gives a greater probability for proton escape along the poles, and a smaller probability along the equator, the Coulomb barrier being smaller and greater, respectively, than is the case for a spherical nucleus. We illustrate this in the three following cases.

(I). $|211+\rangle$, $\frac{3}{2}+ \to |312+\rangle$, $\frac{5}{2}-$

The parent nucleus is the excited Mg_{12}^{24}, which after absorption of the photon has $I_i = 1$, $M_i = \pm 1$, $I_f = \frac{3}{2}$, $K_f = -\frac{3}{2}$. We have

$$\varphi_0(\vartheta', \varphi') = |312+\rangle = \frac{\alpha}{\sqrt{\pi}}\sqrt{\frac{\beta}{\sqrt{\pi}}}\sqrt{2}(\alpha\rho)^2\beta z e^{-\frac{1}{2}(\alpha^2\rho^2+\beta^2 z^2)} e^{i2\varphi'}|+\rangle,$$

where

$$\alpha = \sqrt{\frac{m\omega_\perp}{\hbar}}, \qquad \beta = \sqrt{\frac{m\omega_z}{\hbar}}.$$

Our aim is to see qualitatively how the angular distribution of the emitted protons depends on the behaviour of φ_0 on the surface. We can therefore approximate the functions by neglecting the relatively small difference between α and β. While the sequence of these wave functions gives the well-known very good agreement with experiments, their functional form may be less accurate.

Under these conditions we can approximate

$$|312+\rangle \approx f(r)Y_{32}(\vartheta', \varphi')|+\rangle.$$

The component of the angular momentum along the axis of the nucleus is $\Omega = \frac{5}{2}$.

We consider only the lowest energy state of the daughter nucleus. From $I_1+I_f \geqq j$, we find $j \leqq \frac{5}{2}$. As $\Omega = \frac{5}{2}$, the only possible j is $\frac{5}{2}$.

We insert in the formula for $g_j(r)$ the values of φ_0 and the orthonormal $\varphi_\Omega{}^j$ for the proton, given by

$$\varphi_\Omega{}^j = \sum Y_m{}^1 \chi_\mu{}^{\frac{1}{2}} (1\,\tfrac{1}{2}\, m\mu | j\Omega).$$

The result is

$$g_{\frac{1}{2},\frac{5}{2}}(r) = F(r)(3\,\tfrac{1}{2}\,2\,\tfrac{1}{2}|\tfrac{5}{2}\,\tfrac{5}{2}) \int\int Y_2{}^3 e^{BP_2(\cos\vartheta')} Y_2{}^3 \, d\Omega.$$

We denote the integral by $K_{22}^{33}(B)$.

Transforming to the space fixed system we get

$$f_j(r) - f_{\frac{5}{2}}(r) = (1\,\tfrac{5}{2}\,1 - \tfrac{5}{2}|\tfrac{3}{2} - \tfrac{3}{2}) g_{\frac{1}{2},\frac{5}{2}}(r) = F(r) k_1 k_2 K_{22}^{33}(B).$$

The angular distribution contains only one term. We get

$$P(\vartheta) = v(\tfrac{5}{2}, \tfrac{5}{2}) \left[Y_0{}^0 - \frac{2}{5\sqrt{5}} Y_0{}^2 \right], \qquad \frac{\sigma(0)}{\sigma(\frac{1}{2}\pi)} = 0.5.$$

There is a concentration in the plane perpendicular to the laboratory z-axis. The angular momenta combine to give an alignment of the nuclear axis along the laboratory z-axis defined by the direction of the photons. The concentration of the emitted protons to the plane perpendicular to this direction comes from the properties of the intrinsic wave function $|312\rangle$, which has its maximum at 55°. The non-spherical Coulomb field has in this calculation no influence on the angular distribution, but only on the amplitude. For a light nucleus like Mg^{24} the corrections to the angular distribution from higher j are negligible because of the smallness of g_{l+2}/g_l. The B-dependent factor in the amplitude is the square of $K_{22}^{33}(B)$. For $B = 0.263$ we find $K_{22}^{33}(B) = 1.003$. Thus, the enhancement for the deformed nucleus compared to a spherical nucleus is only 1.006. The reason that the effect is much reduced is that $\varphi_0(\vartheta', \varphi')$ has a maximum where $\exp[BP_2(\cos\vartheta')] = 1$.

(II.) $\quad |202+\rangle, \frac{5}{2}+ \rightarrow |303+\rangle, \frac{7}{2}-$

According to the literal shell-model which we have so far followed, the state $|202+\rangle$ is originally not occupied by a pair. However, even for a nucleus such as Si_{14}^{28} containing two more protons this state would be occupied and the following calculations are valid with only a small numerical change of the factor $F(r)$, provided the coupling scheme appropriate to strongly deformed nuclei is applicable to Si^{28}. Furthermore, when taking into account the short-range interparticle forces, the level in question is occupied by a pair part of the time. The transition considered here is therefore of interest also for Mg^{24}.

The wave function of the excited proton is strongly concentrated at the equator:

$$|303+\rangle = \frac{\alpha}{\sqrt{\pi}}\sqrt{\frac{\beta}{\sqrt{\pi}}}\,\frac{(\alpha\rho)^3}{\sqrt{6}}\,e^{-\frac{1}{2}(\alpha^2\rho^2+\beta^2z^2)}e^{3i\varphi'}|+\rangle.$$

Under the same circumstances as in the previous case this function can be approximated at the surface by

$$|303+\rangle \approx f(r)Y_3{}^3|+\rangle.$$

Thus, we get $\Omega = \frac{7}{2}$. The wave function being separated into a radial and an angular part, we need not discuss the joining of the inner potential and the Coulomb potential.

The daughter nucleus is characterized by the remaining proton from the broken-up pair, in the state $|20-2-\rangle$. Thus one has $I_f = \frac{5}{2}$, $K_f = -\frac{5}{2}$. As $I_1 = 1$, the quantum number is determined by $\frac{3}{2} \leqq j \leqq \frac{7}{2}$ and by $\Omega = \frac{7}{2} \leqq j$, giving $j = \frac{7}{2}$. Therefore, we have now

$$g_{\frac{7}{2},\frac{7}{2}}(r) = F(r)(3\tfrac{1}{2}\,3\tfrac{1}{2}|\tfrac{7}{2}\tfrac{7}{2})K_{33}^{33}(B),$$
$$f_j(r) = (1\tfrac{7}{2}\,1-\tfrac{7}{2}|\tfrac{5}{2}-\tfrac{5}{2})g_j(r).$$

The angular distribution is

$$P(\vartheta) = v(\tfrac{7}{2},\tfrac{7}{2})\left[Y_0{}^0-\frac{\sqrt{5}}{14}Y_0{}^2\right], \qquad \sigma(0)/\sigma(\tfrac{1}{2}\pi) = 0.55.$$

This distribution is analogous to that of the previous case. Though the deformation of the nucleus favours a maximum at the poles, the distribution of $\varphi_0(\vartheta', \varphi')$ on the surface and the coupling of the angular momenta gives a maximum perpendicular to the direction of the photons.

We calculate $K_{33}^{33}(B) = 0.917$. Therefore the decay probability, 0,84, is smaller than for a spherical nucleus. This is explained by $\varphi_0(\vartheta', \varphi')$ being concentrated at the equator, where the Coulomb barrier is greater for a deformed than for a spherical nucleus.

(III). $|220+\rangle, \frac{1}{2}+ \to |330+\rangle, \frac{1}{2}-$

In this case a proton is lifted from a low-lying pair in $|220\rangle$. The odd proton in the daughter nucleus is $|220-\rangle$, or $I_f = \frac{1}{2}$, $K_f = -\frac{1}{2}$, Further, we have

$$|330+\rangle = \frac{\alpha}{\sqrt{\pi}}\sqrt{\frac{\beta}{\sqrt{\pi}}}\,[(\beta z)^3-\tfrac{3}{2}\beta z]e^{-\frac{1}{2}(\alpha^2\rho^2+\beta^2z^2)}|+\rangle.$$

This can be approximated by

$$|330\rangle \approx k_1Y_0{}^1+k_3Y_0{}^3, \qquad k_1 = k(R^3-\tfrac{5}{2}R), \qquad k_3 = k\frac{2}{\sqrt{21}}R^3.$$

When absorbing the photon, the nucleus acquires a spin $I_1 = 1$. Since $K_1 = 0$, all angular momentum is associated with the collective rotation, i.e. $\bar{R} = \bar{I}$. We study the energetically most favoured case, in which the penetrating proton leaves the daughter nucleus in the lowest state of energy, namely minimum collective rotation.

As $I_1 = 1$ and $I_f = \frac{1}{2}$, we have $j = \frac{1}{2}$ and $\frac{3}{2}$, which gives $l = 1$. Further we have $\Omega = \frac{1}{2}$. The result is now

$$g_{\frac{1}{2},\frac{1}{2}}(r) = F_1(r)[k_1 K_{00}^{11}(B) + k_3 K_{00}^{31}(B)](1\,\tfrac{1}{2}\,0\,\tfrac{1}{2}|\tfrac{1}{2}\,\tfrac{1}{2})$$
$$= -F_1(r)K(B)\cdot\frac{1}{\sqrt{3}},$$
$$g_{\frac{1}{2},\frac{1}{2}}(r) = F_1(r)K(B)\cdot\sqrt{\tfrac{2}{3}}.$$

We find that with $k_1 \approx k_3$ and B not much larger than 1, the first term $k_1 Y_0^1$ at the surface gives by far the strongest contribution, so that the term $k_3 Y_0^3$ is unimportant for the penetration of the proton:

$$f_{\frac{1}{2}}(r) = g_{\frac{1}{2}}(r)(1\,\tfrac{1}{2}\,0{-}\tfrac{1}{2}|\tfrac{1}{2}{-}\tfrac{1}{2}) = -F_1(r)K(B)\cdot\tfrac{1}{3},$$
$$f_{\frac{3}{2}}(r) = g_{\frac{3}{2}}(r)(1\,\tfrac{3}{2}\,0{-}\tfrac{1}{2}|\tfrac{1}{2}{-}\tfrac{1}{2}) = -F_1(r)K(B)\cdot\tfrac{1}{3}\sqrt{2}.$$

We find for the angular distribution:

$$P(\vartheta) = v\left[(\tfrac{1}{2}\,\tfrac{1}{2})Y_0^0 + (\tfrac{3}{2},\tfrac{3}{2})\left(Y_0^0 - \frac{1}{2\sqrt{5}}Y_0^2\right) - \{(\tfrac{1}{2}\,\tfrac{3}{2}) + (\tfrac{3}{2}\,\tfrac{1}{2})\}\frac{Y_0^2}{\sqrt{10}}\right]$$
$$= \frac{vF_1^2K^2(B)}{3\sqrt{4\pi}}\left[Y_0^0 - \frac{1}{\sqrt{5}}Y_0^2\right].$$

This gives

$$\sigma(0)/\sigma(\tfrac{1}{2}\pi) = 0.$$

Because of the collective rotation the axis of the nucleus tends to lie in the plane $\vartheta = \frac{1}{2}\pi$, which is perpendicular to the photons. Thus the strong maximum of $\varphi_0(\vartheta', \varphi')$ at the poles of the nucleus explains the property of the angular momentum to be strongly concentrated to this plane.

We calculate that $K_{00}^{11}(B) = 1.117$. Thus, the enhancement for the deformed nucleus is 1.25, due to the fact that $\varphi_0(\vartheta', \varphi')$ in this case has a maximum at the poles, where the escape probability has a maximum.

3.2. THE REACTION $Np_{93}^{234*} \rightarrow U_{92}^{233} + p$

The nuclei hitherto considered have been polarized only by the angular momenta $I_1 = 1$, $M_1 = \pm 1$, coming from the absorbed photon. Then, because of the quantal uncertainty the axis of the nucleus is only weakly directed towards the z-axis in space. We now want a photo-excited nucleus and a daughter nucleus with high spins, and aligned so that their axes point with only small deviations from the z-axis.

We choose the odd nucleus Np_{93}^{234}, in which in the ground state the odd proton occupies $|642+\rangle$, $\frac{5}{2}+$, and the odd neutron $|633-\rangle$, $\frac{5}{2}+$. The photon is supposed to excite the odd proton to the level $|752+\rangle$, $\frac{5}{2}-$, from where it penetrates the barrier. We assume for an isomeric state $I_1 = 5$, and a complete alignment, $M_1 = 5$. For the daughter nucleus, one has $I_f = \frac{5}{2}$ and $K_f = -\frac{5}{2}$. The wave function is now

$$|752+\rangle = \frac{\alpha}{\sqrt{\pi}}\sqrt{\frac{\beta}{\sqrt{\pi}}}\,\frac{1}{\sqrt{15}}\,(\alpha\rho)^2[(\beta z)^5-5(\beta z)^3+\tfrac{15}{4}(\beta z)]e^{-\frac{1}{2}(\alpha^2\rho^2+\beta^2 z^2)}e^{i2\varphi'}.$$

We make the approximation $\alpha = \beta$, and obtain, if the coordinates R are expressed in units of length such that we have $\alpha = 1$,

$$|752\rangle = k\left[1.04\left(\tfrac{4}{99}R^2-\tfrac{4}{9}+\frac{1}{R^2}\right)Y_2{}^3+0.208(\tfrac{2}{13}R^2-1)Y_2{}^5+0.013R^2Y_2{}^7\right]R^5e^{-\frac{1}{2}R^2}.$$

In the previous cases the wave function contained only one $Y_m{}^l$ and the angular distribution was thus independent of R. But in this case we have to determine the boundary between the inner oscillator potential and the outer Coulomb potential. The classical turning point for the proton is given by

$$E = (N+\tfrac{3}{2})\hbar\omega = \tfrac{1}{2}R^2\hbar\omega,$$

for $N = 6$, $R \approx \sqrt{15}$. Putting $R = 4$, we get

$$|752\rangle = \sum_{l'} a_{l'}\,Y_2{}^{l'} = C(0.26Y_2{}^3+0.30Y_2{}^5+0.20Y_2{}^7).$$

We are interested in the qualitative behaviour of $\varphi_0(\vartheta', \varphi')$, and therefore smaller changes in the coefficients of $Y_2{}^l$ may be regarded as unimportant. We therefore accept the above figures.

The quantum number L can take on all even values from 0 to 10. It turns out from the calculations that the coefficients for $L \leqq 4$ are the most important, and we restrict ourselves to them, though some higher coefficients play some role for the test of the positiveness of $P(\vartheta)$. The triangle condition gives $\frac{5}{2} \leqq j \leqq \frac{15}{2}$. This gives $3 \leqq l \leqq 7$.

We first determine $g_{j\Omega}(r)$ for $\Omega = \frac{5}{2}$:

$$g_{j\Omega}(r) = F_1(r)\iint d\Omega \sum_{l'} a_{l'}\,Y_2{}^{l'}|+\rangle e^{BP_2(\cos\vartheta')}\varphi_\Omega{}^{j*}$$

$$= F_l(r)(1\,\tfrac{1}{2}\,2\,\tfrac{1}{2}|j\,\tfrac{5}{2})\sum_{j'} a_{l'}K_{2\,\frac{1}{2}}^{l'\,l}(B).$$

For the case that we consider, B is equal to 0.444, and the integrals $K_{2\,\frac{1}{2}}^{l'\,l}$ are small for $l' \neq l$. In first approximation one has $K_{2\,\frac{1}{2}}^{l'\,l} = \delta_{l'\,l}$. The values of $K_{2\,\frac{1}{2}}^{l'\,l}$ for $B = 0.444$ are given in table 2. As for $F_1(r)$, we have found that $|F_5/F_3| = 0.184$ and $|F_7/F_5| = 0.087$. We put $|F_1(r)| = \alpha_{13}|F_3(r)|$. In the

expression for the angular distribution, it may be possible to neglect higher values of l. Further, with the notations

$$b_j = (5j0-\tfrac{5}{2}|\tfrac{5}{2}-\tfrac{5}{2})(1\,\tfrac{1}{2}\,2\tfrac{1}{2}|j\,\tfrac{5}{2}), \quad A_l(B) = \sum_{l'} a_{l'} K^{l'\,l}_{2\,2}(B),$$

we get

$$f_j(r) = \alpha_{13}|F_3(r)|e^{-i\varphi_1} b_j A_l(B).$$

TABLE 2

Calculated values of $K^{l'\,l}_{2\,2}$ for $B = 0.444$

l' \ l	3	5	7
3	1.009	0.140	0.011
5		1.093	0.166
7			1.063

We calculate the angular distribution $P(\vartheta)$, neglecting terms higher than $L = 4$, which are small. We introduce the notation $n^{(L)}_{jj'}$ by writing the terms in the expression for $P(\vartheta)$ in the form $v(jj')n^{(L)}_{jj'} Y_0{}^L$ and give in table 3 the

TABLE 3

Values of $n^{(2)}_{jj'}$ and $n^{(4)}_{jj'}$

j \ j'	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{15}{2}$
$\frac{5}{2}$	−0.639 0.143	−0.168 0.205	0.174 −0.213	0 −0.043	0 0.024	0 0
$\frac{7}{2}$		−0.479 −0.052	−0.136 0.119	0.186 −0.219	0 −0.032	0 0.017
$\frac{9}{2}$			−0.469 −0.061	−0.107 0.100	0.157 −0.201	0 −0.017
$\frac{11}{2}$				−0.530 −0.004	−0.082 0.094	0.102 −0.150
$\frac{13}{2}$					−0.635 0.116	−0.056 0.082
$\frac{15}{2}$						−0.774 0.320

coefficients $n^{(2)}_{jj'}$ and $n^{(4)}_{jj'}$ with the former in the upper line. The coefficients are symmetrical in j and j'. We obtain for the angular distribution

$$P(\vartheta) = v\frac{|F_3(r)|^2}{\sqrt{4\pi}} \sum_{jj'} (Y_0{}^0\delta_{jj'} + Y_0{}^2 n^{(2)}_{jj'} + Y_0{}^4 n^{(4)}_{jj'}) \times \alpha_{13}\alpha_{l'3} b_j b_{j'} A_l A_{l'} e^{i(\varphi_l - \varphi_{l'})}.$$

Using the notation

$$P(\vartheta) = \sum_L Y_0^L \sum_{l \leqslant l'} k_{ll'}^{(L)} \times 10^{-2} A_l A_{l'},$$

we obtain the values of table 4, with $k^{(0)}$, $k^{(2)}$, $k^{(4)}$ ordered vertically. Further we calculate A_l for $B = 0.444$ and compare them in table 5 with a_l for $B = 0$.

TABLE 4

Calculated values of $k^{(0)}$, $k^{(2)}$ and $k^{(4)}$

l' \ l	3	5	7
3	2.82 −1.47 −0.017	0 −0.081 0.112	0 0 0.0003
5		0.573 −0.345 0.037	0 −0.0028 0.0046
7			0.0016 −0.0012 0.00038

We calculate $P(\vartheta)$ for these values. For $B = 0$ we obtain

$$P(\vartheta) = C[Y_0^0 - 0.561 Y_0^2 + 0.046 Y_0^4].$$

TABLE 5

Calculated values of A_l and a_l

l	3	5	7
a_l	0.26	0.30	0.20
A_l	0.31	0.40	0.27

For $B = 0.444$ we get

$$P(\vartheta) = C1.5[Y_0^0 - 0.570 Y_0^2 + 0.052 Y_0^4].$$

Thus we obtain an enhancement with the factor 1.5.

4. Semi-Classical Limit: I_1 and I_t Large Compared to j

In the previous cases the quantal uncertainty in the direction of the axis of the nucleus has influenced the angular distribution of the emitted protons. We now approach the classical case of a fixed axis by examining the limit of large I_1 and I_t, thus getting the influence of only the distribution $\varphi_0(\vartheta', \varphi')$ on the surface of the nucleus.

To this end we take the asymptotic values of the angular distribution for $L = 2$. We have

$$\begin{pmatrix} I_1 & I_1 & 2 \\ -M_1 & M_1 & 0 \end{pmatrix} \to (-1)^{I_1+M_1} \cdot \frac{3M_1^2/I_1^2-1}{2\sqrt{2I_1}}.$$

Further we calculate

$$\begin{Bmatrix} I_1 & I_1 & 2 \\ j & j' & I_t \end{Bmatrix}$$

for the possible values $j' = j, j+1, j+2$. The limiting values can be obtained from the exact formulae which are given e.g. by Edmonds [5]). For instance, one has

$$\begin{Bmatrix} I_1 & I_1 & 2 \\ j & j & I_t \end{Bmatrix} \to \frac{(-1)^{I_1+I_t+j} 2[3(I_1-I_t)^2-j(j+1)]}{\sqrt{2I_1}\sqrt{(2j-1)2j(2j+1)(2j+2)(2j+3)}}.$$

In the limit $I_1 \to \infty$ we thus find a finite result for $P(\vartheta)$.

We calculate the angular distribution of the protons as a function of $\varphi_0(\vartheta', \varphi')$ in this limiting case, when the axis of the nucleus can keep a fixed position in space. We illustrate this by a case treated above, namely case (III), in subsect 3.1. where $\varphi_0(\vartheta', \varphi') = |330+\rangle$, which has a maximum at the poles and is zero at the equator. Because of the influence of the coupling of the angular momenta $I_1 = 1$, $I_t = \frac{1}{2}$, we found there a maximum not along the z-axis of the system fixed in space but instead in the xy-plane. In this case j was determined by $\Omega = \frac{1}{2}$ and the condition $j \leqq I_1 + I_t$ to be $\frac{1}{2}$ and $\frac{3}{2}$. Also for large I_1 and I_t we can limit ourselves to these values of j, as $F_l(r)$ can be neglected for $l \geqq 3$. For $g_j(r)$, which is independent of I_1 and I_t, we get the same values as in case (III). Further, one has

$$f_j(r) = (-1)^{I_t-I_1+\Omega}(I_1, j, K_t+\Omega, -\Omega|I_t K_t) g_{j\Omega}(r),$$

where we have to calculate the limit. We find

$$f_{\frac{1}{2}}(r) \to \sqrt{\frac{I_t+K_t+1}{2I_t}} g_{\frac{1}{2}\frac{1}{2}}(r), \qquad f_{\frac{3}{2}}(r) \to \sqrt{\frac{I_t+K_t+1}{2I_t}} \frac{(3K_t-I_t)}{2I_t} g_{\frac{3}{2}\frac{1}{2}}(r).$$

We calculate $P(\vartheta)$ for the case that $K_t = I_t$, $I_1 = I_t + \frac{1}{2}$, and find

$$P(\vartheta) = v \frac{F_1(r)^2 K(B)^2}{\sqrt{4\pi}} \left[Y_0^0 + \frac{3M_1^2/I_1^2-1}{\sqrt{5}} Y_0^2 \right].$$

For $M_1 = I_1$ the value of $P(\vartheta)$ is maximum for $\vartheta = 0$ and zero for $\frac{1}{2}\pi$. For $M_1 = 0$ the situation is the inverse. Thus we clearly see the influence of the value of the wave function on the surface of the proton, which is zero at the equator and strong at the poles.

I thank Dr. Sven A. E. Johansson and Dr. Sven Gösta Nilsson for very valuable discussions on different aspects of these calculations.

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10.E *Nuclear Physics* **28** (1961) 681; © *North-Holland Publishing Co., Amsterdam*
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ERRATA

S. N. Biswas and V. Gupta, Pion-hyperon resonances in pion production in hyperon-nucleon collisions, Nuclear Physics **24** (1961) 620.

In sect. 2 the expressions for σ_1 and σ_3, in terms of the amplitudes, should be interchanged. In table 1, read $\sigma_1 = \frac{1}{3}|\frac{2}{5}\sqrt{5}\Gamma_{33}-\Gamma_{31}|^2$ and $\sigma_3 = \frac{3}{5}|\Gamma_{33}|^2$ and interchange σ_1 and σ_3 in (5) and (6). Finally, the value of R_0 in the first row of table 3 should be $\frac{2}{3}$.

We should like to thank Miss T. K. Radha and Miss R. Thunga for drawing our attention to this error.

H. Fröhlich, The structure of momentum space, the neutrino and the Pauli principle, Nuclear Physics **26** (1961) 324.

Line 6 of sect. 4.3 should read: "of (x_k, x_0) by $(-x_k, -x_0)$ . . ." instead of: "of(x_k, x_0) by $(-x_k, x_0)$ . . .".

10.A *Nuclear Physics* **28** (1961) 682—684; © *North-Holland Publishing Co., Amsterdam*

NEWS ITEMS

CERN ANNUAL REPORT FOR 1960

A new lay-out has been adopted for this 115 page illustrated report which is divided into nine chapters. A historical summary recalls the aims of CERN and its programme; it gives some key dates in the life of the Organization since the entry into force of the interim agreement between eleven signatory States, in 1952, and shows by what stages the staff grew from 30 in 1953 to 1052 by the end of 1960.

To meet the requirements of the "operational" stage, twelve divisions have now replaced the six original divisions responsible for the "construction" of the main research facilities, viz, the big particle accelerators. The report reviews the progress of these six original divisions in 1960.

The Theory Division has two major tasks at CERN: one is to carry out fundamental research in the various branches of theory and the other is to provide theoretical support to the experimental groups of the laboratory.

The 28 GeV proton synchrotron was gradually made available to experimental research teams, while its performance was steadily improved throughout the year, particularly the intensity of the accelerated proton beam. The report also gives an account of the construction and operation of track chambers at CERN and indicates the part played in 1960, in the exploitation of the big accelerator, by the chamber from the Ecole Polytechnique, Paris.

As to the *600 MeV synchro-cyclotron*, it continued to provide facilities for many nuclear physics experiments, while part of its own experimental teams began to perform experiments with the synchrotron. The measurement of the anomalous magnetic moment of the μ-meson and the completion of a μ-meson channel are two of the many subjects dealt with in this chapter.

The Scientific and Technical Services concentrated mainly on improving instruments for the evaluation of bubble chamber photographs, on developing automation processes for them, and on electronic computing.

Finally the report reviews the work done in 1960 by the *Site and Buildings Division* and quotes a few suggestive figures, notably the electric power consumption, which rose to 23.5 million kWh in 1960 and accounted for 1/28 of the total consumption of the Geneva network.

More figures can also be found in the part of the report dealing with *Administration*, particularly with respect to staff, expenditure, budget appropriations, etc.

In his introduction the Director-General of CERN, John B. Adams, discusses remerkable prospects of future development of the laboratory.

Finally, with regard to external relations, the report explains the provisions made to secure an efficient co-operation between visiting scientists and the staff of CERN and also mentions the exchanges of scientific staff with the Joint Institute for Nuclear Research at Dubna, USSR.

PROFESSOR V. F. WEISSKOPF, DIRECTOR-GENERAL OF CERN

Prof. Victor F. Weisskopf, who joined CERN in September last year as member of the Directorate responsible for research, became fourth Director-General of the Organization on 1 August 1961. He is succeeding Dr. J. B. Adams who returns to the United Kingdom as director of the Controlled Thermonuclear Research Establishment, at Culham.

Victor Frederick Weisskopf was born in Vienna on 19 September 1908. He studied under Max Born and James Franck at the University of Göttingen, where he obtained his doctorate in physics in 1931. His thesis was devoted to the "fundamental theory of the interactions of light with atoms".

In 1931 Victor Weisskopf began his travels round the Universities, which helped to make him "a citizen of the world", a scientist freed from the restriction of frontiers and such conventional concepts. Weisskopf was therefore in Leipzig in 1931, acting as an assistant to Heisenberg, and a year later he was in Berlin as assistant to Schrödinger. He spent his summers at the Institute for Theoretical Physics directed by Niels Bohr in Copenhagen.

During the early 1930's the difficult situation developed in Central Europe resulted in the exile of many thousands of Europeans. Victor Weisskopf was also drawn into the migratory movement. During the latter part of 1932 he accompanied his colleague Placzek to the Institute of Physics and Technology at Kharkov in the USSR. There they both took part in theoretical research under the direction of L. Landau.

Weisskopf returned to the West in 1933. Until 1935 he was assistant to Wolfgang Pauli at the Federal Polytechnic Institute (ETH) at Zurich. He then rejoined Niels Bohr in Copenhagen to take up an appointment as research associate, working on the theory of radiation and nuclear structure.

Then, as many other European physicists, Victor Weisskopf joined in the move accross the Atlantic: at the end of 1937 he accepted an appointment as professor at the University of Rochester. He remained there until 1943, when he went once more West, to Los Alamos, New Mexico. Working there on applications of nuclear energy, he was to become "the oracle" for his colleagues.

In October 1945, Victor Weisskopf was offered an appointment to teach physics at the Massachusetts Institute of Technology in New England. V. Weisskopf taught at M.I.T. until 1960 and at the same time supervised research on nuclear and elementary particles structure. During the past 15 years he has been a frequent visitor to Europe. In 1950 and 1951 he was guest professor at Sorbonne University and at the Zurich Polytechnic Institute where, 20 years before, he had done research with Pauli.

Prof. Weisskopf first came to Meyrin on 9 August 1957 as a guest professor. He stayed for a year with the Theoretical Studies Division and took part in research work on the 600 MeV synchro-cyclotron which CERN was then putting into operation. He was therefore no stranger to CERN matters when his name was proposed as a member of the directorate in charge of research. Victor Weiskopf who always encouraged international co-operation, accepted his appointment. M.I.T. granted him a leave of absence and on 12 September he returned to Meyring.

Victor Weiskopf's main interest is theoretical physics and he has published a number of articles on this subject and on philosophical matters. However, his major work is a 900 page book, *Theoretical Nuclear Physics*, written in co-operation with John M. Blatt in 1952 and translated into Russian.

Prof. Weisskopf is a corresponding member of the French Academy of Science, a member of the National Academy of Science in the U.S. and until the end of 1960, he was President of the American Society of Physics. In recognition of his work he was awarded the Planck medal by the German Society of Physics in 1960.

CERN Public Information Office

FIRST JOHN T. TATE MEDAL OF AMERICAN INSTITUTE OF PHYSICS to Dr. Paul Rosbaud

Dr. Paul Rosbaud, European scientific editor who has made distinguished contributions to publications in physics, is the first recipient of the John T. Tate International Medal for Distinguished Service to the Profession of Physics. On 28 September 1961, in New York, Dean Ralph A. Sawyer of the University of Michigan, Chairman of the Governing Board of the American Institute of Physics, presented him a gold medal and a $1000 award "for his stimulating and unconquerable devotion to conscientious publication in the physical sciences."

The Tate Medal has been established in order to recognize distinguished service to physics by individuals who are neither residents nor citizens of the United States. The late John Torrence Tate, who died on May 27, 1950, was Dean of the College of Science, Literature, and Arts at the University of Minnesota, and served as Research Professor of Physics, Chairman of the Governing Board of the American Institute of Physics (A.I.P), and Managing Editor of the American Physical Society.

Dr. Rosbaud before and during the last war was science editor for the Springer Company, a Berlin publishing house. As an editor, he is credited with being the driving force behind the creation of new journals and the improvement of existing periodicals, as well as being responsible for the publication of many books by scientists. One of his recent achievements was the preparation of the volume in honour of Wolfgang Pauli, "Theoretical Physics in the Twentieth Century", published by Interscience Publishers, New York. He suggested the publication of the collected works of the famous British physicist, Lord Rutherford. The first volume will be issued in the near future.

Born in Graz, Austria in 1896, Dr. Rosbaud received his doctorate in Berlin and did research in X-ray crystallography at one of the Kaiser Wilhelm Institutes. He joined the Springer publishing house in the 1930's and succeeded in sending its valuable scientific books to safety before the bombings. His extraordinary courage and acquaintanceship with friends in high places, who were uncommitted to the Nazi regime, enabled him to keep in touch with scientists in occupied Europe during the war and to boost their morale. He had two French war prisoners, who were physicists, staying with him on parole, and enabled them to work and keep in touch with scientific developments. They are now prominent atomic scientists. During the war he risked his life and freedom by communicating with Allied scientists via the Norwegian underground and neutral channels and by being the confidant of German physicists who opposed the decline of science under the Nazis.

After the war he was brought to England by the Allies and he became editor for Butterworth Scientific Publications, and later served as Scientific Director for Pergamon Press. Since 1956 he has been European Editor for Interscience Publishers and a consultant to other publishing companies. He succeeded in obtaining manuscripts for publication from Russian scientists at a time when this was very difficult, stimulated the international exchange of scientific information, and suggested to the London "Times" a scientific supplement for the intelligent layman, a suggestion later brought to reality.

10.D *Nuclear Physics* **28** (1961) 685—694; © *North-Holland Publishing Co., Amsterdam*

BOOK REVIEWS

Proceedings of the Second Conference on reactions between complex nuclei, Gatlinburg, Tennessee, edited by A. ZUCHER, E. C. HALBERT and F. T. HOWARD (John Wiley and Sons, New York, 1961. ix-319 p. 56 s.)

Proceedings of the International Symposium on polarization phenomena of nucleons, edited by R. HUBER and K. P. MEYER (Birkhäuser Verlag, Basel, 1961. 436 p. S. Fr. 65.)

Another two conferences out of the many — another two volumes of proceedings for the library shelves. There is very little synthesis in these volumes: they are just collections of papers grouped around a limited theme. Their appeal is thus limited to the specialists working in the domains in question; but for these specialists their value is very great, owing to the high quality of the contributions. We have here good examples of the one type of really useful conferences: meetings of specialists confronting their methods and results, and even (as examplified by the "Basle † convention") agreeing to bring into their language some measure of unification. (The other type of useful conference is that in which broad synthetic surveys are attempted, as was done so successfully at Seattle in 1956).

To say that the topics of the two conferences are highly specialized does not imply any underrating of their importance: both deal with phenomena likely to provide decisive clues to a deeper understanding of nuclear forces. The study of reactions between complex nuclear systems has perhaps the added appeal of lending itself to semi-classical considerations of great elegance (such as Ford and Wheeler's "rainbow" scattering picture). The proceedings of the Gatlinburg conference have the rare merit of keeping for posterity a faithful record, not only of the physicists' early fumblings with these new phenomena, but also of their exhilaration and sense of humour.

L. R.

† Not "Basel convention"! Apparently nobody in the venerable centre of humanism (celebrating, in fact, its 500th anniversary) was aware of the fact that it is part of its fame (like Ghent, Lyons and some other old cities) to have its name misspelt as "Basle" in English.

Theoretical Physics in the Twentieth Century, A Memorial Volume to Wolfgang Pauli, edited by M. FIERZ and V. F. WEISSKOPF (Interscience Publishers, New York, 1960. x-328 p. $ 10.00).

This volume, which was meant to be offered to Pauli on his sixtieth birthday, comes to appear as a tribute to his memory. It is a worthy testimonial to the many-sided creative and critical work of the great physicist, who left his mark in every field of fundamental research of contemporary physics. In spite of the care bestowed by the editors in planning the volume and selecting the contributors among Pauli's friends and most distinguished pupils, the record is still incomplete—it does not touch upon Pauli's deep interest in the epistemological aspects of quantum theory, or his attribution of the hyperfine structure of atomic spectra to a nuclear spin †, which deserve more than foot-notes in the bibliography.

The articles which compose the volume are intended to be historical surveys of the problems, but they differ very much both in length and in scope; some are brief assessments of the state of physics, other longer reviews, other still contain personal recollections and extracts of correspondence, which are of great historical value. All make most interesting and inspiring reading.

L. R.

† See on this point S. Goudsmit's spirited article in *Physics to-day* **14**, No. 6 (1961) 18, which could have been a most fitting contribution to the memorial volume.

P. ROMAN, *Theory of elementary particles* (second edition) (North-Holland Publ. Co., Amsterdam, 1961, xxvi–580 p. fl. 50)

The didactic excellence of this treatise, which was pointed out in the review of the first edition in this journal (**16** (1960) 189), explains the necessity of issuing a second edition within a year. Apart from the correction of minor slips and a few additions, the new edition does not differ from the first. The author, in a new preface, states his reasons for leaving out of consideration the method of dispersion relations; they are not very convincing. Still, I think he has been wise to limit the scope of his book in the way he has done. The student who has mastered its contents is well equipped for proceeding to the more elaborate techniques of field theory with the help of the masterly textbooks from the school of Bogolyubov.

L. R.

L. R. B. ELTON, *Nuclear Sizes* (Clarendon Press, Oxford University Press, London, 1961. 114 p. 15 s.)

In this valuable monograph an authoritative survey is given of the various phenomena which provide information about the density distribution of nuclear matter and nuclear force within the nucleus and of the theoretical attempts at a schematic description of the characteristics of the nuclear "surface layer".

The discussion of fast electron scattering is unfortunately vitiated by the author's failure to understand the logical point of the argumentation based on Bodmer's and Reignier's analytical method, which deserves better treatment than is meted out to it (Reignier's work is not even quoted). The author's own analysis of the data, while not incorrect, is of uncertain physical significance, since it relies too much on the central density, a parameter which (as he himself correctly points out) is not accurately determined by the electron scattering data.

L. R.

S. K. FRIEDLANDER and L. TOPPER (Editors), *Turbulence: Classical papers on statistical theory* (Interscience Publishers, London, 1960. ix–187 p. $6.00)

This is a useful collection of reprints of fundamental papers, by G. I. Taylor, v. Kármán, Kolmogoroff and others, on the statistical theory of turbulence. The dates of publication range from 1921 to 1949, and the editors add a bibliographical list of more recent papers on various aspects of this important subject.

L. R.

WOLFGANG PAULI, *Aufsätze und Vorträge über Physik und Erkenntnistheorie* (Friedr. Vieweg & Sohen, Braunschweig, 1960. vii-183 p. DM 19.80.)

In this book are collected Pauli's writings on general or historical aspects of particular physical questions and on the epistemological problems raised by the development of modern physics. Apart from four † translations from the English, the articles are here reprinted in the original German. We are thus enabled to enjoy Pauli's very characteristic style, which (especially for those who have known Pauli) is such a faithful reflection of his personality. Pauli himself took great pains in the arrangement of the book, which now, after his untimely death, may be regarded as his philosophical testament. It thus deserves to be studied with special care, for Pauli's thinking on philosophical problems (in contrast to what could be said of some other great physicists) was characterized by uncommon openmindedness, depth and earnestness.

L. R.

† The publisher's note stating that three articles are translated is inaccurate, since the paper on Einstein's contribution to quantum theory was also originally written in English.

N. N. Bogolyubov, B. V. Medvedev and M. K. Polivanov, *Theory of dispersion relations*, UCRL Trans. 499 (translated by S. G. Brush) (Lawrence Radiation Laboratory, 1961, 198 p.)

Since the publication of the original monograph in 1958, a large number of papers have been published in which the scope of the method of dispersion relations has been variously extended; moreover, a fresh approach to the whole problem has been initiated by Mandelstam and is still vigorously pursued. Notwithstanding this process, the masterly treatment due to Bogolyubov and his collaborators has lost nothing of its importance and usefulness. It contains, in particular, a treatment of generalized functions and functions of several complex variables, used in Bogolyubov's famous proof of the dispersion relations for the scattering of π-mesons on nucleons, as well as a discussion of the spectral representation of Green's functions. The present translation must therefore be hailed as a welcome addition to the literature. It is issued in typescript form, the displayed formulae, however, being directly reproduced from the printed original.

L. R.

Nelson M. Cooke and John Markus, *Electronics and Nucleonics Dictionary* (McGraw-Hill Book Co., London, 1960. ii-543 p. 93 s.)

How can it be possible to review a dictionary? Do you have to read the whole book through and control each of the more than 12000 definitions (I have not counted them, but you can read that number on the cover)? I cannot believe that the reviewer should have to do that job. I have taken all the words I could think of in my own field of interest and I have noticed first that it was possible to find the definition of these words in the dictionary. Secondly, the definitions were short and correct and supported by clear illustrations in cases of more complicated questions.

Such a book is indeed very valuable if you could get people who write technical articles to use it so that they use the technical words in their original meaning. And it would be a great advantage if you could force salesmen to use technical words in the sharp definitions given by this dictionary instead of using these words for materials they do not cover or instead of constructing new words for matters which are already covered by generally accepted words.

One can only praise the authors of this book for the tedious job they have been doing and it is hoped that the book will be widely used as a handbook in the precise definitions of technical words.

J. Thomas

Niels Bohr, *Physique atomique et connaissance humaine* (traduit par E. Bauer et R. Omnès) (Gauthiers-Villars, Paris, 1961. 98 p. 15 NF.)

Cette collection d'articles et d'essais fait suite à celle dont la traduction française a été publiée par le même éditeur en 1932 sous le titre *Théorie atomique et description des phénomènes*. On peut y suivre l'élaboration de la conception de complémentarité, tant par son extension à tous les domaines de la connaissance, que par la formulation de plus en plus précise de l'idée fondamentale. Il convient avant tout de s'arrêter à l'essai intitulé „Discussion avec Einstein sur des problèmes épistémologiques de la physique atomique", qui par l'allure historique de l'exposé éclaire singulièrement l'évolution de la question et le caractère inéluctable de la solution à laquelle Bohr aboutit. Quant aux aspects complémentaires de la biologie, il importe de signaler que les articles qui y ont trait dans le présent recueil représentent un stade de la pensée de l'auteur qu'il a maintenant dépassé.

Les particularités du style de Bohr, qui rendent la lecture de ses écrits d'une difficulté quasi-proverbiale, n'ont pas rebuté les éminents traducteurs: le professeur Bauer et son collaborateur R. Omnès ont réussi à transposer les longues périodes de l'original dans un français très pur qui en rend parfaitement l'esprit.

L. R.

M. A. SCHULTZ, *Control of Nuclear Reactors and Power Plants* (McGraw Hill, London 1961. 2nd ed. ix–462 p. 97 s.)

It is quite interesting for a physicist to read this bible of nuclear reactor control. Just like some of the reactor physics texts are the basic sources of principles of reactor physics, this book is the basic source of reactor control philosophy. Nevertheless, the language used by a reactor physicist and the language used by a control engineer are so different that these two kinds of nuclear engineers have to do a lot of translation in order to understand each other. The author contributes to this translation problem by describing nuclear reactors to the control engineer.

After an introductory chapter follows a chapter on elementary physics of reactor control, then a chapter on elementary servo-mechanisms which gives an excellent introduction to the concept of a transfer function. After these two chapters reactor kinetics is described and a transfer function is introduced for the nuclear reactor considered as a control element. It is realized that the complete set of reactor kinetics equations constitutes a set of non linear equations, but for the purpose of control investigations this set of equations is linearized and then transfer functions are derived for fast and slow reactors and for U, Th and Pu as fuel. In the beginning the delayed neutrons are treated and later on more and more problems are taken into account like temperature coefficients, effects of poison, effects of voids.

By means of the Nyquist stability criterion the different types of reactor and control stability conditions are derived and considered with respect to control criteria.

Chapter 6 is devoted to automatic reactor control with much emphasis on transient behaviour. Chapter 7 describes the reactor control mechanisms i.e. control rods, liquid poison, liquid level, spectral shift. The control rod system including various drive mechanisms is described in most detail.

Chapter 8 discusses nuclear power plant control where the heat exchanger, turbine, condenser and external load are incorporated in the control discussion. Chapter 9 deals with the special problems encountered with boiling reactor plants.

Chapter 10 deals with radiation detectors for reactor control. This chapter is the only one where I might have a slightly critical remark: very little emphasis is placed on the statistical nature of nuclear phenomena. Whereas this is unimportant for most reactor operations it might be considered important for critical experiments and for the lowest power levels during reactor start up which is dealt with in chapter 11.

Chapter 12 discusses power operation and chapter 13 deals with the shutdown procedure. Chapter 14 might be considered as an appendix dealing with analogue computing techniques and its application to plant simulators.

The book is furnished with an excellent collection of references. It is an engineering book, but even so, a book that reactor physicists should know about.

O. Kofoed-Hansen

E. GLUECKAUF (Editor), *Atomic Energy Waste: Its Nature, Use and Disposal* (Interscience Publishers Inc.,; Butterworths, London. 1961. xi-420 p. 95 s.)

During the coming five years in Britain, less than one per cent of the total funds spent on atomic power stations will be required for treatment and storage facilities of waste products. This information is given in the editor's introduction to the present book on atomic energy waste. To-day, contrary to the early years of atomic energy, most people dealing with atomic energy waste feel that the problems concerning the diposal of waste are not as serious and insurmountable as previously assumed.

As indicated in the sub-title, the six parts of the book treat of the nature, use and disposal of the waste products. They are entitled: The waste materials of atomic energy; Effects of radiation on materials and living organisms; Legal aspects of atomic energy waste disposal; Operations involved in the treatment of radioactive waste; Fission products as sources of radiation; Uses of radiation in research and agriculture. The different parts are written by well-known British authorities on the special subjects treated.

As a reviewer it is difficult to judge the treatment of so many different items with which this

book is concerned. Only dealing with those items with which I am familiar, I find the presentation excellent. Perhaps some matters are rather elementarily treated; on the other hand many of the other items in the book would seem rather advanced to me. So all in all E. Glueckauf has succeeded in editing a most valuable book, which has something to say to all people concerned with atomic energy waste.

A special interest centers perhaps on the problems concerned with the treatment and disposal of high activity waste. Quantitative removal of caesium and strontium causes a reduction of toxicity in the remainder of 20 year old fission products by a factor of about 25000 and, it should thus be possible to dispose of the remainder safely in the sea after about 15 years' storage, but in future reactors, when high burn-up and recycling of plutonium occurs, 470 y Am^{241} will be produced in quantity, e.g. from reaction of neutrons on U^{238}, and this means that the toxicity of the remainder after removal of strontium and caesium will then be governed by the quantity of Am^{241} present. As tank storage is costly one has tried to find new ways of waste disposal. In the chapter on Conversion of highly active wastes to solid products, C. B. Amphlett discusses the new and very interesting aspects of the problem. One of the most promising methods seems to be a melting process of the solutions containing fission products with nepheline syenite and calcium hydroxide. Using this method one gets a glass with a density of about 2 and with a very good resistance to leaching. The costs of waste processing and storage using the nepheline syenite process are estimated to amount to about 1% of the kWh (E) price.

Asker Årkrog

Codes for Reactor Computations, Proceedings of a Seminar, Vienna, 25—29 April 1960 (I.A.E.A,. Vienna, 1961. 538 p. $ 8.00)

Apparently IAEA had at least three reasons for last year's Vienna Seminar on codes for reactor computations: firstly, to encourage atomic energy institutions to go in heavily for electronic digital computers by demonstrating their extreme usefulness in the field of reactor computations; secondly, to give experienced physicists and mathematicians an opportunity to interchange knowledge within this field; finally, to investigate the possibility of forming a library of computing codes formulated in a universal machine language. In fact, in the last three or four years before 1960, most reactor calculations were done on one or two machines, and it was rather easy to interchange reactor codes; but the introduction in 1960 of a large number of transistor machines presents major problems with respect to reactor codes, as it is nearly certain that from now on a variety of machines will be used in the reactor computation field.

The book has a somewhat incidental content of papers of high as well as of low standard. Presumably the attempt of IAEA to cover these many objects in the symposium is one of the reasons for this.

The book contains 36 papers, printed in the original language (English, in four cases French) with abstracts in the four languages of IAEA. The papers are divided in the following subject groups:

A. Facilities for reactor computations;
B. Utilization of computers for reactor problems;
C. Code libraries, existing and planned;
D. Universal languages;
E. Universal and statistical methods.

This grouping does not seem to be quite adequate. Some of the papers really belong to two or three of the groups and some are not placed in the right category. The subjects of the papers vary from surveys of reactor codes to papers dedicated to special physical and numerical topics. Mostly the papers include references. The discussions seem to be reported correctly, only few mistakes or omissions are noticed.

As a whole, a more limited scope of the symposium and consequently also of the proceedings, would have been advantageous. The book will nevertheless be useful to beginners as well as to advanced workers in the field of reactor computations, especially because it is the first of its kind and because it reviews and evaluates many of the reports published on these topics.

In connection with the preparation of reactor codes, a considerable amount of work has been done on numerical methods, for instance on the solution of partial differential equations and the computation of higher order approximations in transport theory. Therefore the book can be of some use also to physicists within other fields of physics, and to numerical analysts.

H. Vilstrup

R. G. PALMER and A. PLATT, *Fast Reactors*, Nuclear Engineering Monographs No. 12 (Temple Press, London 1961. x-93 p. 12 s. 6d.)

Again an excellent addition to this useful little series of monographs on nuclear engineering.

It is briefly stated that the world's energy resources of uranium and thorium can only be considered large compared with fossile fuel reserves if breeding ratios larger than one can be achieved and that the one case in which it has been established that such a breeding ratio can be obtained is the fast reactor, although the $U^{233}-Th^{232}$ fuel cycle shows some promise also. This brief statement, however, places the fast reactor with all its complications in the proper light and this point ought to have a much more widespread publicity than what seems to be the case.

Chapter one gives an excellent introduction to the fast reactor concept and enumerates its major technological problems.

Chapter two describes the choice of materials for fast reactors both as regards fuels, coolants, canning and structural materials. Chapter three then describes liquid sodium and sodium-potassium technology since this field is sufficiently important to warrant a complete chapter.

Chapters four and five deal with fast reactor physics and the subject is divided into what is termed statics and dynamics although more appropriate headlines would probably be steady state and kinetics.

Finally, chapter six treats heat transfer problems in liquid metals. This chapter might thus just as well have appeared after chapter three, but this is a minor detail.

All the nuclear engineering monographs are intentionally rather short. This calls for a certain amount of 'do it yourself' manipulations when reading them for the first time. After the reader has overcome this hurdle, however, he will find that the present book is very valuable and highly commendable.

O. Kofoed-Hansen

DOUGLAS S. BILLINGTON and JAMES H. CRAWFORD, *Radiation damage in solids* (Princeton University Press, Princeton, 1961. xi—450 p. $ 12.50)

I was sitting at my desk preparing a series of lectures on Radiation Damage when the book arrived and my first reaction was: "how wonderful, here I have all the material that I need". And this is indeed so, the book contains an enormous amount of facts and an enormous list of references and quotations of opinions on the subject. However, as dealt with in this book the subject resembles zoology or some such fairly empirical discipline and very little is done in order to systematize the many empirical facts. Thus it becomes a fairly difficult subject to extract essentials from the text. This is probably very good because the subject is so new and so much remains to be done that at present nobody knows what is essential and what is unessential.

The book may be considered to fall into three sections. In the first section we find the background knowledge necessary for work in radiation damage. This material is covered in a chapter on interaction between radiation and the crystalline state, a chapter on the influence of lattice imperfections on the physical properties of solids and finally a chapter describing the different radiation sources such as reactors, accelerators etc.

The second section contains results of radiation damage studies on different groups of solids. Thus there are chapters on radiation effects in metals, alloys, covalent crystals and mineral structures, ionic crystals and finally semiconductors.

The third and last section deals with radiation damage to uranium and to graphite. These two materials are given special chapters because the technological interest in these materials has warranted the collection of a very comprehensive amount of information on them.

The subject of radiation damage is very large and one could write books on individual subjects like damage to graphite. Thus it may seem silly when I am going to say that something is missing in the book, for of course the subject is so big that many items must be missing if a single volume shall be enough. However, I shall endeavour to say that the addition of two chapters would have been useful.

One of these should have been a chapter on dosimetry. Most radiation damage experiments yield curves of "damage" (that means relative change of a physical parameter) as a function of dose and, consequently, the accuracy of such radiation damage experiments is as dependent on damage as on dose. Especially in nuclear reactors dose is often more difficult to define and to measure than damage. The authors are fully aware of this point as can be seen from the chapter on radiation sources where dosimetry is briefly mentioned and it is a pity that they do not devote more pages to this subject.

The second chapter that I miss is a chapter on the kinetics of radiation damage. The kinetics of radiation damage is not understood in any great detail yet. Fully developed it must describe the growth of damage with dose, dose rate, and thermal treatment, i.e. it must include all aspects of damage production and annealing that can be described by relatively simple equations giving the number of defects as a function of time. As already mentioned our knowledge of this subject is limited, but enough ideas have appeared in the literature to warrant a special chapter on the subject which is of considerable interest both from a fundamental and a technological point of view.

In spite of the above critical remarks it should again be emphasized that the book contains a wealth of information and will undoubtedly prove of considerable use.

O. Kofoed-Hansen

Michel BAYET, *Physique Nucléaire* (Masson et Cie, Paris, 1960. 404 pages. 65NF).

Comme le laisse supposer la généralité de son titre, l'ouvrage de M. Bayet est essentiellement un manuel d'enseignement. Il peut avantageusement servir de base à un cours donné au niveau de la licence (deuxième cycle). A l'exemple de nombre d'ouvrages écrits en anglais au même niveau, l'auteur présente d'abord quelques résultats élémentaires de physique théorique sur les moments angulaires et les spins, les statistiques et quelques notions de relativité.

Le rayonnement cosmique et la physique des „hautes énergies" sont exclus du corps de l'ouvrage qui est consacré exclusivement à la physique des „basses énergies". Chaque chapitre constitue en soi un exposé suffisamment développé pour que l'étudiant puisse y acquérir les notions quantitatives qui doivent constituer la base d'enseignements ultérieurs plus spécialisés. Quelques notices ou remarques historiques agrémentent souvent la lecture de cet ouvrage d'enseignement dont on peut cependant regretter qu'il ne contienne pas un seul problème.

J. Humblet

W. R. HARPER, *Basic principles of fission reactors* (Interscience Publishers, Inc., New York, 1961, viii—314 $ 7.50).

This book is intended as an introductory text to be used for postgraduate courses, but should also be useful for the young student or for the engineer or scientist well acquainted with other topics, but wanting an introduction to nuclear reactor principles.

The book is somewhat unorthodox in its treatment of the subject and undoubtedly one may disagree with the form of the presentation in many points of detail. However, the book is indeed fulfilling a need since it presents an interesting amount of information not usually contained in reactor physics texts or in reactor engineering books and dealt with from basic points of view.

Historical facts are treated by the choice of examples. Thus a chapter is devoted to describing the principle of a nuclear reactor by describing Fermi's first "voyage to the new continent" and thus introducing the concept of criticality by description of the first criticality ever made. In this way and by letting the reader get to the drawings of power reactors during the first few chapters the author succeeds in arriving at a very impressive setting in which he now describes the fission

process essentials and the fuel element including radiation damage to fuel. Then follows a chapter on neutron physics and thus equipped the reader can embark upon a description of the fast reactor which is illustrated by a description of the bare or should we say naked "Lady Godiva". The fast reactor is compared to the bomb and Figs. 7.7 and 7.8 show explosion damage caused by Godiva and by a bomb illustrating the very significant difference between reactors and bombs. After this description of fast reactors the moderator is described and thermal reactors follow, first homogeneous reactors, next heterogeneous reactors. Then comes a chapter describing some of the important aspects behind the philosophy of reactor design. The chapter is called "The Influence of Purpose on Design" and the reader finds among other items an estimate of the world's power needs in this chapter. The final five chapters deal with the control of radioactivity, radiation protection, cooling, reactor kinetics and reactor operation. The entire text is kept at the level where very specialized mathematics is avoided to a large extent and the book appears much more as a "philosophy" of nuclear power than as a mathematical description of nuclear reactors. The possible dangers encountered in nuclear energy are dealt with in an admiringly frank and direct fashion which does not exaggerate neither minimizes and consequently is completely free of the atomic hysteria so frequently encountered.

The fact that the presentation is unorthodox both in sequence and in selection makes me feel that readers may find the book odd; however, many valuable and pertinent points are clearly considered in this text and not readily found in the usual texts.

O. Kofoed-Hansen

LAURENCE H. ALLER, *The abundance of the elements* (Interscience Publishers, London, 1961. 1961. xi-283 p. $ 10.00.)

In this monograph a very thorough survey is given of the distribution of the elements in the earth, the meteorites, the gaseous nebulae, the stellar atmospheres and the cosmic radiation; this is followed by a discussion of the relative abundance of isotopes and the composition differences between stars; finally, the theories of the origin of the elements, which have recently received such remarkable elucidation, are summarized. The style of exposition is that of the usual reviews of recent literature, which is not meant to be a praise. There is in the book an enormous amount of diligently compiled information, but it is offered to the reader with an apparent aloofness which is rather disheartening. The specialist in astrophysics will no doubt appreciate the practical usefulness of such a compilation, but others will regret that so fascinating a story is not more attractively told.

L. R.

A. ABRAGAM, *The principles of nuclear magnetism* (Clarendon Press: Oxford University Press, London, 1961. xvi-599 p. 84 s.)

I reckon to my most pleasing remembrances of the post-war period the Amsterdam conference on spectroscopy at radio-frequencies in 1950. Apart from my old acquaintance with F. Bloch, this was my first contact with the physicists engaged in research in the domain of nuclear magnetism, and I was struck by the difference between their general attitude of mind, their "style" (as Pauli used to say), and that of the people working in the more central fields of nuclear physics. While the latter feel the strain of unrelenting effort and the uncertainties of adventurous gropings, and become more and more absorbed in collective activities dominated by machinery of the most complex character, the former are in the fortunate position of following a line of remarkably harmonious development in which theory and experiment concur by contributions sufficiently simple for the individual to understand them in detail, while retaining a grasp of the whole. With this in mind it is perhaps easier to realize how such a rare masterpiece of elegance as Prof. Abragam's treatise could make its appearance among the shabby crowd of hasty and fragmentary reports which more and more goes to form the so-called "literature" of nuclear physics.

Nuclear magnetism derives its peculiar simplicity and fundamental unity behind a diversity

of phenomena from the smallness of the nuclear magnetic moments, which implies that the radiative processes involving them belong to a domain of frequencies allowing of classical theoretical description as well as of refined techniques of production and analysis. Especially the occurrence in various experimental circumstances of sharp resonances, which are produced by simple means and susceptible to clear-cut interpretation, opens the way to a deep-going investigation of high accuracy both of nuclear properties and of many aspects of the structure of matter. The book under review presents for the first time a connected and systematic account of the principles underlying the resonance methods and a detailed survey of their applications. The author has taken great pains in re-thinking all the arguments and disposing them, with admirable insight and skill, in the best possible order. The result is not only the much-needed fundamental treatise, to which every one will refer for initiation, orientation or inspiration, but at the same time a work of art in the finest tradition.

L. R.

J. THEWLIS, Editor-in-Chief, *Encyclopaedic Dictionary of Physics*, Vol. 1 (Pergamon Press, Oxford, 1961. xv-800 p. £80 per set).

It may well be questioned whether the effort put into the production of an encyclopaedic dictionary of physics is at all worth while. This question arose already when the new Springer Encyclopedia began to appear, and it is much more acute when one is faced, not with a more or less systematic presentation as in the latter case, but with the random collection of separate fragments which the alphabetical order of succession produces.

Whatever view one takes about this question, it is at any rate a matter of satisfaction to know that the work is in good hands: this first volume exhibits a high standard of precision and reliability in the single articles, and shows that the Editor-in-chief has been guided by sound criteria in his difficult task of selecting the articles to be written and deciding the approximate length of each of them. The one point in which I think he has been ill-inspired is in adopting Clarendon type for the standard symbols of physical constants, and in not imposing the uniform system of units recommended by the SUN Commission of the IUPAP.

L. R.

J. BASS, *Cours de mathématiques*, Tomes I et II (Masson et Cie, Editeurs, Paris, 1961. xxii-576 p. 440 p. 50 NF, 38 NF.)

Cours écrit à l'intention des élèves des grandes écoles d'ingénieurs. Il offre une revue rapide des sujets suivants: (1) algèbre linéaire (matrices et tenseurs), (2) ensembles, fonctions, intégrales simples, (3) fonctions définies par séries ou intégrales, (4) intégrales curvilignes et intégration mécanique, (5) surfaces et intégrales multiples, (6) fonctions analytiques, (7) équations différentielles (y compris calcul symbolique et résolution numérique), (8) équations aux dérivées partielles, potentiels, fonctions harmoniques, (9) calcul des variations, abaques, algèbre de Boole.

L. R.

N. E. BREZHNEVA and S. N. OZIRANER, *Radioactive isotopes and their production under neutron irradiation*, Review Series No. 15 (International Atomic Energy Agency, Wien, 1961. 70 p. 6s.)

Although not the most spectacular result of the introduction of nuclear energy to the benefit of mankind, the availability of radioactive isotopes certainly is the most important from economical and practical points of view.

The majority of radioactive isotopes in current use is produced in nuclear reactors, and this review describes the processes and methods utilized for the preparation of pure radioisotopes from irradiated material.

The description falls naturally into three parts corresponding to the three main types of nuclear

reactions, neutron capture, (n, α) and (n, p) reactions and fission. Detailed examples are given in each group, illustrating the implications pertaining to this kind of production, as well as the various chemical and physical methods employed to obtain a satisfactory product.

The book is heartily recommended to all those not familiar with this interesting mixture of nuclear physics and radiochemistry.

The full text is given in English, as well as in Russian.

K. Heydorn

International Conference on theoretical aspects of very high energy phenomena (CERN, Geneva, 1961. v-426 p).

This specialists' conference was held at CERN from the 5th to 9th June, 1961 and the very neatly produced offset issued of the proceedings is dated 11th August: we have here one more example of CERN's remarkable efficiency. In such a still largely unexplored field, prompt publication of progress reports is especially useful, and it is gratifying to see that CERN is able to fulfil this requirement in such a brillant fashion.

Altogether, this conference was timely and well-planned, and apart from their immediate utility to the workers in the field, its proceedings will also enable other physicists to get a precise idea of the encouraging progress so far achieved as well as of the formidable difficulties which tend to slow it down. In spite of the emphasis on theoretical aspects, there is in the volume plenty of material on planned and performed experiments, and on the whole evidence of a most healthy symbiosis of the two types of physicists whose co-ordinated efforts are felt to be more and more indispensable as one penentrates into less and less accessible regions.

The reports are for practical reasons classified under eight sections, but the underlying logical subdivision of the whole field is the semi-empirical characterization of the interactions as weak, intermediate (electromagnetic) and strong. The latter have of cource the lion's share (sections III to VII): after an excellent survey of the experimental evidence from all available sources, the various theoretical approaches (statistical, diffraction, one-meson exchange, dispersion relations) are in turn discussed, with renewed consideration of appropriate experimental material. The last section is devoted to further accelerator projects, and the volume closes with a masterly summary of the situation by L. Van Hove. All through the apparently sober reporting of modest results and cautious prospects, one feels quite a refreshing breeze of adventure and optimism which augurs well for the future.

L.R.

J. F. Loutit and R. Scott Russel (Editors), Progress in Nuclear Energy, Series VI, Biological Sciences, vol. 3: *The entry of fission products into food chains* (Pergamon Press, Oxford, 1961. 174 p. 45s).

This volume gives an account of investigations on the movement of fission products into food chains which were undertaken in connection with nuclear bomb testings at Maralinga (Operation Buffalo). The results, written as 8 reports by different authors, are summarized and discussed in a concluding chapter by the editors. Among the main topics are retention of fallout by vegetation, the solubility of fission products and their metabolism by animals.

The amount of fission products retained by vegetation varied with the plant species, particle size of fallout deposit and climatic conditions. The solubility of fallout samples from the ground surface and from the radioactive clouds indicated a relatively high biological availability. The experiments with cows, rabbits and sheep elucidated that iodine isotopes form the major sources of hazard in the early stages after the deposition of fallout, whilst, at a later period the bone-seekings isotopes of strontium become dominant. Although other fission products such as tellurium and molybdenum are metabolized to some extent, they seem to be of much less hazard than iodine and strontium isotopes owing to their short physical and biological half-lives.

The book should be of interest not only to biologists but also to Civil Defence and Public Health Authorities.

A. Andersen

AUTHOR INDEX

BARKER, F. C., Positive parity states of Be^{9} and C^{13} 96

BARNARD, A. C. L. and C. C. KIM, Elastic scattering and reactions of protons on argon-40. 428

BARNARD, A. C. L., G. S. MANI and P. D. FORSYTH, Gross properties of levels in A^{38} with excitation near neutron separation energies . . 464

BARNARD, A. C. L., *see* KIM

BARNARD, A. C. L., *see* MANI

BASS, R., H. P. HÄENNI, T. W. BONNER and F. GABBARD, Disintegration of K^{39} by fast neutrons 478

BEARD, G. B. and W. H. KELLY, "Self-scintillation" study of the beta decay of Rb^{87} 570

BÈS, D. R. and Z. SZYMAŃSKI, The ground-state equilibrium deformation in rare-earth nuclei 42

BONNER, T. W., *see* BASS

DE BOUARD, X., H. LANGEVIN-JOLIOT, N. MARTY et B. TATISCHEFF, Excitation du niveau de 15.1 MeV de C^{12} par diffusion inélastique des protons de 150 MeV 293

BRUNS, C. A., *see* ZDANIS

BURDE, J. and M. RAKAVY, Determinations of the lifetime of the 283 keV excited level of Dy^{160} 172

CALVERT, J. M., *see* TAKAYANAGI

CARLSON, RICHARD R., The 9.63 MeV state of C^{12} 443

CAROVILLANO, ROBERT L., *see* WILDERMUTH

CHURCH, E. L. and J. WENESER, Anomalous electric-dipole internal conversion 602

CUPERMAN, SAMI, Search for pseudoscalar interaction in $\frac{1}{2}+ \rightarrow \frac{1}{2}-$ β-decay transitions. (I) Measurement of the longitudinal polarization of the β^{-}-particles emitted in the transition $_{81}Tl^{207}$ $_{82}Pb^{207}$ 84

DEO, BIBHUTI BHUSAN, Electromagnetic interactions of charged pions 122

DEO, BIBHUTI BHUSAN, On masses and mass-shifts of elementary particles 135

DEVARE, H.G., On the decay of I^{132} 148

DUBBELDAM, P. S. and R. L. WALTER, Polarization of D-D neutrons for deuteron energies from 1.9 to 11 MeV 414

EDWARDS, S., *see* ZDANIS

EL NADI, M. and H. SHERIF, Stripping of heavy projectiles. 331

EWAN, G. T., *see* GEIGER
FORSYTH, P. D., *see* BARNARD
GABBARD, F., *see* BASS
GALE, N. H., *see* TAKAYANAGI
GARG, J. B., *see* TAKAYANAGI
GEIGER, J. S., R. L. GRAHAM and G. T. EWAN, The decay of Ce^{144} . . . 387
GRAHAM, R. L., *see* GEIGER
GUMAN, V. N., YU. I. KHARITONOV, L. A. SLIV and G. A. SOGOMONOVA, Pairing forces and pair correlations in Pb^{206}, Pb^{210} and Po^{210} . . . 192
GUSTAFSON, TORSTEN, Angular distribution of photo-protons from deformed nuclei 665
HÄENNI, H. P., *see* BASS
HANS, H. S., *see* KHURANA
HANSEN, O., Energy levels in the stable isotopes of titanium 140
HILLION, PIERRE and JEAN-PIERRE VIGIER, Interaction theory in a new isobaric spin space 624
HÖGAASEN-FELDMAN, J., A study of some approximations of the pairing force 258
HOLLANDER, JACK M., *see* HULTBERG
HOREN, DANIEL J., *see* HULTBERG
HULTBERG, SÖLVE, DANIEL J. HOREN and JACK M. HOLLANDER, Measurements of internal conversion coefficients 471
IONIDES, P. A., General equivalence theorem in vector field theory . . 662
JARMIE, NELSON, Concering the existence of rare light nuclei 426
JOHNSTON, ROBERT R. and KENNETH M. WATSON, The second-order optical potential for pion-nucleus scattering 583
KAMEFUCHI, S., L. O'RAIFEARTAIGH and ABDUS SALAM, Change of variables and equivalence theorems in quantum field theories . . . 529
KELLY, W. H., *see* BEARD
KHARITONOV, YU. I., L. A. SLIV and G. A. SOGOMONOVA, Pairing forces and pair correlation in Tl^{206} and Bi^{210} 210
KHARITONOV, YU. I., *see* GUMAN
KHURANA, C. S. and H. S. HANS, Cross-sections for (n, 2n) reactions at 14.8 MeV 560
KIM, C. C. and A. C. L. BARNARD, Elastic scattering of protons by argon-36 438
KIM, C. C., *see* BARNARD
KÖNIG, L. A., J. H. E. MATTAUCH and A. H. WAPSTRA, Adjustment of relative nuclidic masses (II) $70 < A < 200$ 1
KRÓLIKOWSKI, W., On single-strange mesons 81
KRUEGER, T. K. and B. MARGOLIS, The s-wave and p-wave strength functions 578

LANGEVIN-JOLIOT, H., *see* DE BOUARD
MADANSKY, L., *see* ZDANIS
MANI, G. S., A. C. L. BARNARD, T. A. TOMBRELLO and D. A. A. S. N. RAO, Levels in A^{38} observed through the Cl^{37} (p, n)A^{37} reaction near threshold 456
MANI, G. S., *see* BARNARD
MARGOLIS, B., Two-photon decay and the lifetime of giant resonance states 524
MARGOLIS, B., *see* KRUEGER
MARTY, N., *see* DE BOUARD
MATTAUCH, J. H. E., *see* KÖNIG
MEYER, M. A., Multiple Coulomb scattering from finite nuclei 512
MÖBIUS, P., Different modes of motion in special translationally invariant many-body systems 304
NÉMETH, J., The effect of pairing energy on the temperature excitation energy of an interacting fermion system 407
NOVIKOV, V. M., *see* ZARETSKI
O'RAIFEARTAIGH, L., *see* KAMEFUCHI
OWEN, G. E., *see* ZDANIS
PHILLIPS, R. J. N., On the analysis of the reaction $\pi^- + p \to \Lambda^0 + K^0$. 519
PRANGE, R. E., The moment of inertia of large systems at finite temperature 369
PRANGE, R. E., The moment of inertia of superfluid fermion systems of finite temperature. 376
PRATT, WILLIAM W., A survey of gamma ray branching ratios 598
RAKAVY, M., *see* BURDE
RAO, B. N. SUBBA, On the levels and transitions in Gd^{155} 503
RAO, D. A. A. S. N., *see* MANI
RAYNAL, J., Diffusion de protons polarisés par une cible de protons polarisés 220
SALAM, ABDUS, *see* KAMEFUCHI
SAWICKI, JERZY and TOSHIO SODA, Theory of collective excitations of spherical nuclei with particular application to the giant dipole state 270
SHAFROTH, S. M., The nucleus Y^{88} and the Y^{88}—Sr^{88} mass difference . . 649
SHAPIRO, I. S., On the dispersion theory of direct nuclear reactions . . 244
SHERIF, H., *see* EL NADI
SODA, TOSHIO, *see* SAWICKI
SLIV, L. A., *see* GUMAN
SLIV, L. A., *see* KHARITONOV
SOGOMONOVA, G. A., *see* GUMAN
SOGOMONOVA, G.A., *see* KHARITONOV
STRUTINSKI, V. M., Angular correlations in statistical nuclear reactions 342

SZYMAŃSKI, ZDZISŁAW, Equlibrium deformations of nuclei in the transuranic region 63
SZYMAŃSKI, Z., *see* BÈS
TAKAYANAGI, S., N. H. GALE, J. B. GARG and J. M. CALVERT, Neutron threshold measurements in the reactions $^{9}Be(p, n)^{9}B$, $^{10}B(p, n)^{10}C$, $^{11}B(p, n)^{11}C$ 494
TATISCHEFF, B., *see* DE BOUARD
TOMBRELLO, T. A., *see* MANI
TOMOZAWA, YUKIO, Higher resonances in pion-nucleon reactions and the isobar model 160
VIGIER, JEAN-PIERRE, *see* HILLION
WALTER, R. L., *see* DUBBELDAM
WAPSTRA, A. H., Adjustment of relative nuclidic masses (III) $A \geqq 200$ 29
WAPSTRA, A. H., *see* KÖNIG
WATSON, KENNETH M., *see* JOHNSTON
WENESER, J., *see* CHURCH
WERNER, E., Density modulated ground states in a gas of interacting fermions 353
WILDERMUTH, KARL and ROBERT L. CAROVILLANO, Optical giant resonances in nuclear reactions 636
ZARETSKI, D. F. and V. M. NOVIKOV, Theory of nuclear excitations by muons in heavy mesic atoms 177
ZDANIS, R. A., C. A. BRUNS, G. E. OWEN, L. MADANSKY and S. EDWARDS, Exchange stripping effects in the reaction $B^{11}(d, n)\ C^{12}_{4.43}$ 550

SUBJECT INDEX

1. Nuclear Structure

1.B *Two- and three-body problems*

Diffusion de protons polarisés par une cible de protons polarisés, J. Raynal . . 220

1.C *Many-particle systems*

Pairing forces and pair correlations in Pb^{206}, Pb^{210} and Po^{210}, V. N. Guman, Yu. I. Kharitonov, L. A. Sliv and G. A. Sogomonova 192
Pairing forces and pair correlation in Tl^{206} and Bi^{210}, Yu. I. Kharitonov, L. A. Sliv and G. A. Sogomonova. 210
A study of some approximations of the pairing force, J. Högaasen-Feldman . 258
Different modes of motion in special translationally invariant many-body systems, P. Möbius. 304
Density modulated ground states in a gas of interacting fermions, E. Werner 353
The moment of inertia of large systems at finite temperature, R. E. Prange. . 369
The moment of inertia of superfluid fermion systems of finite temperature, R. E. Prange 376
The effect of pairing energy on the temperature excitation energy of an interacting fermion system, J. Németh. 407

1.D *Nuclear models*

1.D.1 *Shell model*

Positive parity states of Be^{9} and C^{13}, F. C. Barker 96
Theory of collective excitations of spherical nuclei with particular application to the giant dipole state, Jerzy Sawicki and Toshio Soda 270
A survey of gamma ray branching ratios, William W. Pratt 598

1.D.2 *Collective model*

The ground-state equilibrium deformation in rare-earth nuclei, D. R. Bès and Z. Szymański 42
Equilibrium deformations of nuclei in the transuranic region, Zdzisław Szymański. 63
Energy levels in the stable isotopes of titanium, O. Hansen 140
Determination of the lifetime of the 283 keV excited level of Dy^{160}, J. Burde and M. Rakavy 172
On the levels and transitions in Gd^{155}, B. N. Subba Rao 503
Angular distribution of photo-protons from deformed nuclei, Torsten Gustafson 665

1.D.3 *Other models*

Optical giant resonances in nuclear reactions, Karl Wildermuth and Robert. L. Carovillano 636

1.D.4 *Relationship between different models*

Paring forces and pair correlation in Pb^{206}, Pb^{210} and Po^{210}, V. N. Guman, Yu. I. Kharitonov, L. A. Sliv and G. A. Sogomonova 192

Pairing forces and pair correlation in Tl^{206} and Bi^{210}, Yu. I. Kharitonov L. A. Sliv and G. A. Sogomonova 210

1.E *Properties of nuclei*

1.E.1 *Energy levels*

Energy levels in the stable isotopes of titanium, O. Hansen 140
On the decay of I^{132}, H. G. Devare 148
The decay of Ce^{144}, J. S. Geiger, R. L. Graham and G. T. Ewan . . . 387
The 9.63 MeV state of C^{12}, Richard R. Carlson 443
Disintegration of K^{39} by fast neutrons, T. Bass, H. P. Häenni, T. W. Bonner and F. Gabbard 478
On the levels and transitions in Gd^{155}, B. N. Subba Rao 503
The nucleus Y^{88} and the $Y^{88}-Sr^{88}$ mass difference, S. M. Shafroth . . 649

1.E.2 *Binding energies; masses*

Adjustment of relative nuclidic masses (II) $70 < A < 200$, L. A. König, J. H. E. Mattauch and A. H. Wapstra 1
Adjustment of relative nuclidic masses (III) $A \geqq 200$, A. H. Wapstra 29
Concerning the existence of rare light nuclei, Nelson Jarmie 426
The nucleus Y^{88} and the $Y^{88}-Sr^{88}$ mass difference, S. M. Shafroth . . 649

1.E.3 *Electric and magnetic moments*

The ground-state equilibrium deformation in rare-earth nuclei, D. R. Bès and Z. Szymański 42
Equilibrium deformations of nuclei in the transuranic region, Zdzisław Szymański 63

1.E.4 *Transition probabilities; cross sections*

Determination of the lifetime of the 283 keV excited level of Dy^{160}, J. Burde and M. Rakavy 172

1.E.6 *Nuclear moments of inertia*

The moment of inertia of large systems at finite temperature, R. E. Prange 369
The moment of inertia of superfluid fermion systems of finite temperature, R. E. Prange 376

2. Nuclear Reactions

2.A *General properties*

2.A.1 *Cross sections, energy relations*

Neutron threshold measurements in the reactions $^{9}Be(p, n)^{9}B$, $^{10}B(p, n)^{10}C$, $^{11}B(p, n)^{11}C$, S. Takayanagi, N. H. Gale, J. B. Garg and J. M. Calvert 494

2.B *Angular distributions; correlations; polarization*

Polarization of D-D neutrons for deuteron energies from 1.9 to 11 MeV, P. S. Dubbeldam and R. L. Walter 414

2.C *Resonance reactions; R-matrix theory*

Elastic scattering and reactions of protons on argon-40, A. C. L. Barnard and C. C. Kim 428

Levels in A^{38} observed through the $Cl^{37}(p, n)A^{37}$ reaction near threshold, G. S. Mani, A. C. L. Barnard, T. A. Tombrello and D. A. A. S. N. Rao . . 456
Gross properties of levels in A^{38} with excitation near neutron separation energies, A. C. L. Barnard, G. S. Mani and P. D. Forsyth 464
Disintegration of K^{39} by fast neutrons, R. Bass, H. P. Häenni, T. W. Bonner and F. Gabbard 478

2.D *Statistical model*

Angular correlations in statistical nuclear reactions, V. M. Strutinski 342
The effect of pairing energy on the temperature excitation energy of an interacting fermion system, J. Németh 407
Cross-sections for (n, 2n) reactions at 14.8 MeV, C. S. Khurana and H. S. Hans 560

2.E *Optical model*

Levels in A^{38} observed through the $Cl^{37}(p, n)A^{37}$ reaction near threshold, G. S. Mani, A. C. L. Barnard, T. A. Tombrello and D. A. A. S. N. Rao. . . . 456
Gross properties of levels in A^{38} with excitation near neutron separation energies, A. C. L. Barnard, G. S. Mani and P. D. Forsyth 464
The s-wave and p-wave strength functions, T. K. Krueger and B. Margolis 578
Optical giant resonances in nuclear reactions, Karl Wildermuth and Robert L. Carovillano 636

2.F *Direction interaction (other than stripping)*

On the dispersion theory of direct nuclear reactions, I. S. Shapiro 244

2.G *Stripping and pick-up*

On the dispersion theory of direct nuclear reactions, I. S. Shapiro 244
Stripping of heavy projectiles, M. El Nadi and H. Sherif 331
Exchange stripping effects in the reaction $B^{11}(d, n)\ C^{12}_{4.43}$, R. A. Zdanis, C. A. Bruns, G. E. Owen, L. Madansky and S. Edwards 550

2.H *Coulomb excitation*

Theory of nuclear excitations by muons in heavy mesic atoms, D. F. Zaretski and V .M. Novikov 177

2.I *Photonuclear procecesses*

Theory of collective excitations of spherical nuclei with particular application to the giant dipole state, Jerzy Sawicki and Toshio Soda 270
Two-photon decay and the lifetime of giant resonance states, B. Margolis . . 524
Angular distribution of photo-protons from deformed nuclei, Torsten Gustafson 665

2.J *Fission and spallation*

Theory of nuclear excitations by muons in heavy mesic atoms, D. F. Zaretski and V. M. Novikov 177

2.L *Scattering processes (except two-nucleon system and meson scattering)*

Energy levels in the stable isotopes of titanium, O. Hansen 140
Excitation du niveau de 15.1 MeV de C^{13} par diffusion inélastique des protons de 150 MeV, X. de Bouard, H. Langevin-Joliot, N. Marty et B. Tatischeff 293
Elastic scattering and reactions of protons on argon-40, A. C. L. Barnard and C. C. Kim. 428
Elastic scattering of protons by argon-36, C. C. Kim and A. C. L. Barnard . . 438

3. Electromagnetic Radiation

3.A *Gamma Radiation*

The decay of Ce^{144}, J. S. Geiger, R. L. Graham and G. T. Ewan 387
The 9.63 MeV state of C^{12}, Richard R. Carlson 443
On the levels and transitions in Gd^{155}, B. N. Subba Rao 503
Two-photon decay and the lifetime of giant resonance states, B. Margolis . . 524
A survey of gamma ray branching ratios, William W. Pratt 598

3.B *Internal conversion*

Determination of the lifetime of the 283 keV excited level of Dy^{160}, J. Burde and M. Rakavy 172
Measurements of internal conversion coefficients, Sölve Hultberg, Daniel J. Horen and Jack M. Hollander 471
Anomalous electric-dipole internal conversion, E. L. Church and J. Weneser . 692

4. Beta-Decay; Parity

4.B *Formulation of Fermi theory*

Search for pseudoscalar interaction in $\frac{1}{2}+ \rightarrow \frac{1}{2}-\beta$-decay transitions. (I) Measurement of the longitudinal polarization of the β^--particles emitted in the transition $_{81}Tl^{207} \rightarrow {}_{82}Pb^{207}$, Sami Cuperman 84

4.E *Analysis of data*

On the decay of I^{132}, H. G. Devare 148
Concerning the existence of rare light nuclei, Nelson Jarmie 426
"Self-scintillation" study of the beta decay of Rb^{87}, G. B. Beard and W. H. Kelly 570

6. Methods and Instruments

6.D *Tables and computational aids*

Adjustment of relative nuclidic masses (II) $70 < A < 200$, L. A. König, J. H. E. Mattauch and A. H. Wapstra 1
Adjustment of relative nuclidic masses (III) $A \geqq 200$, A. H. Wapstra . . . 29

7. Quantum Field Theory

7.A *General*

Change of variables and equivalence theorems in quantum field theories, S. Kamefuchi, L. O'Raifeartaigh and Abdus Salam 529
General equivalence theorem in vector field theory, P. A. Ionides 662

7.C *π-meson field (scattering and nuclear forces)*

Electromagnetic interactions of charged pions, Bibhuti Bhusan Deo 122
Higher resonances in pion-nucleon reactions and the isobar model, Yukio Tomozawa. 160
The second-order optical potential for pion-nucleus scattering, Robert R. Johnston and Kenneth M. Watson 583

8. Elementary Particles

8.B *Theory: interactions, transformations properties*

On single-strange mesons, W. Królikowski 81
On masses and mass-shifts of elementary particles, Bibhuti Bhusan Deo. . . 135
On the analysis of the reaction $\pi^- + p \to \Lambda^0 + K^0$, R. J. N. Phillips 519
Interaction theory in a new isobaric spin space, Pierre Hillion and Jean-Pierre Vigier 624

9. Cosmic Radiation

9.B *Passage of particles through matter*

Multiple Coulomb scattering from finite nuclei, M. A. Meyer 512

10. News Items

10.A *News Items*

CERN Annual Report for 1960 682
Professor V. F. Weisskopf, Director-General of CERN 682
First John T. Tate Medal of American Institute of Physics to Dr. Paul Rosbaud 683

10.D *Book Reviews*

Proceedings of the second conference on reactions between complex nuclei, Gatlingurg, Tennessee (L.R.) 685
Proceedings of the international symposium on polarization phenomena of nucleons (L.R.) 685
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Wolfgang Pauli, Aufsätze und Vorträge über Physik und Erkenntnistheorie (L. R.) 686
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Nelson M. Cooke and John Markus, Electronics and nucleonics dictionary (J. Thomas) 687
Niels Bohr, Physique atomique et connaissance humaine (traduit par E. Bauer et R. Omnès) (L. R.) 687
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International Conference on theoretical aspects of very high energy phenomena (L. R.) 694
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10.E *Errata*

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